PROOF OF A CONJECTURE OF ROUTLEDGE

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According to Routledge [1], every g.r. (general recursive) function of one variable can be expressed as $g(\phi(a))$. Here $\phi(a)$ is a p.r. (primitive recursive) function and $g(a)$ is a function defined by the schema

$$g(n) = m,$$

$$g(a) = h(a, g(\delta(a))), \qquad \text{for } a \neq n$$

where $h(a, b)$ and $\delta(a)$ are p.r. functions and $\delta(a) \prec a$ for $a \neq n$ in a well-ordering, of order type ω, of the natural numbers with n as the first element. Routledge also conjectured that not every g.r. function can be expressed as $\phi(g(a))$ [1].

The purpose of this note is to give a proof for this conjecture by actually constructing a g.r. function $\psi(a)$ and then showing that for any p.r. function $\phi(a)$ and any function $g(a)$ defined by the preceding schema, $\psi(a)$ is not identically equal to $\phi(g(a))$. An argument for which $\psi(a) \neq \phi(g(a))$ can be actually found by using the method in the proof, provided that the well-ordering of type ω involved in the definition of $g(a)$ is itself constructively given. In this sense, the proof can be regarded as a constructive one.

DEFINITION. A finite sequence of natural numbers

$$a_0, a_1, \cdots, a_r$$

is called a w.o. (well-ordered) segment of rank p, if and only if $r > 0$, $a_r = p$, $a_i \neq a_j$ for $i \neq j$, and $a_i \prec a_r$ for $i < r$.

PROOF OF THE CONJECTURE. Let $\varphi_1(y, a)$ and $\varphi_2(y, a, b)$ be two enumerating functions of p.r. functions of one variable and two variables respectively [2]. Let $\tau(a, b, c, d, e)$, $\tau_1(a)$, $\tau_2(a)$, $\tau_3(a)$, $\tau_4(a)$, $\tau_5(a)$ be six p.r. functions such that for any five given numbers a, b, c, d, e, a unique number x exists so that $\tau_1(x) = a$, $\tau_2(x) = b$, $\tau_3(x) = c$, $\tau_4(x) = d$, $\tau_5(x) = e$ and $\tau(a, b, c, d, e) = x$. (These functions can be constructed readily by using the p.r. functions $\sigma(a, b)$, $\sigma_1(a)$, $\sigma_2(a)$ as defined by Peter [3].) Now, let a g.r. function $\psi(a)$ be constructed as follows:

The first step. Put $\psi(0) = 0$.

The $(p+1)$th step $(p > 0)$. There are obviously only a finite number

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of w.o. segments of rank p. Let this finite number be $\pi(p)$. Consider any one of such segments $a_0, a_1, \cdots, a_r$. We shall determine a number x from this segment and then use this number for the evaluation of $\psi(p)$. Let x be the least number y not being used at the previous steps, such that y satisfies the conditions:

(i) $$\tau_1(y) = a_0,$$

(ii) $$\varphi_1(\tau_5(y), a_i) = a_{i-1}, \qquad \text{for } i = 1, \cdots, r.$$

It is noted that this number x can always be found. For, obviously there exists a number u^* such that the p.r. function $\varphi_1(u^*, a)$ has the property $\varphi_1(u^*, a_i) = a_{i-1}$ for $i = 1, \cdots, r$; then the expression $\tau(a_0, l, s, t, u^*)$ with l, s, t as parameters gives infinitely many numbers all satisfying the conditions (i) and (ii) when one of the parameters, say, l runs over all the natural numbers.

We use the number x to define a partial recursive function $k(a)$ by

$$\begin{aligned} k(a) &= \varphi_1(\tau_3(x), g(a)), \\ g(\tau_1(x)) &= \tau_2(x), \\ g(a) &= \varphi_2(\tau_4(x), a, g(\varphi_1(\tau_5(x), a))), \qquad \text{for } a \neq \tau_1(x). \end{aligned}$$

Since $p = a_r$ and x satisfies the conditions (i) and (ii), then it is seen that the value of $k(a)$ for $a = p$ is uniquely determined by the above equations and can be evaluated in a finite number of steps.

Since there are $\pi(p)$ w.o. segments of rank p, then we can determine $\pi(p)$ numbers $x_{p,1}, \cdots, x_{p,\pi}$ and then use them to define $\pi(p)$ partial recursive functions $k_{p,1}(a), \cdots, k_{p,\pi}(a)$ respectively in a similar way as described above. To conclude this step, we let

$$\psi(p) = 1 + k_{p,1}(p) + \cdots + k_{p,\pi}(p).$$

The function $\psi(a)$ as constructed above is effectively calculable for each argument a and consequently $\psi(a)$ is a g.r. function. (See, for example, [4].)

Let $\phi(a)$ be any p.r. function and $g(a)$ be any function defined at the beginning of this note. A number w is called a favorable argument of the function $\phi(g(a))$, if for any number $i > 0$, $\delta^i(w) < w$, if $\delta^i(w) \prec w$. We shall show in the following that there are infinitely many such favorable arguments and among them there is at least one argument for which $\psi(a) \neq \phi(g(a))$.

Given any natural number q, we find the number k such that q is the kth element in the ordering $\prec$. Find the greatest number z among the first k elements in the ordering $\prec$ and again find the greatest number w among the $(k+1)$th, $\cdots$, $(k+z+2)$th elements in the ordering $<$. If $<$, the well-ordering of type ω, is constructively given, the numbers k, z and also w can be actually found. It can be easily verified that $q<w$, $n<w$ and that for any y, $y<w$, if $y\prec w$. Since $\delta(a)\prec a$ for $a\neq n$, then for any $i>0$, $\delta^i(w)<w$, if $\delta^i(w)\prec w$. Consequently w is a favorable argument of $\phi(g(a))$. Since w is greater than the arbitrarily given number q, then the number of such favorable arguments is infinite.

We find three numbers s, t, u, such that $\varphi_1(s, a)=\phi(a)$, $\varphi_2(t, a, b)=h(a, b)$ and $\varphi_1(u, a)=\delta(a)$. Let $x_0=\tau(n, m, s, t, u)$. Suppose w be any favorable argument of $\phi(g(a))$. Since n is the first element in the ordering $\prec$, we can find a number r which is the least number i such that $\delta^i(w)=n$. Then the sequence $\delta^r(w), \cdots, \delta(w), w$ is a w.o. segment of rank w with its initial term $\delta^r(w)=n$. Denote this w.o. segment by $a_0, \cdots, a_r$. Then we have $\tau_1(x_0)=n=a_0$ and $\delta(a_i)=\varphi_1(\tau_5(x_0), a_i)=a_{i-1}$ for $i=1, \cdots, r$. Hence x_0 satisfies the conditions (i) and (ii). According to the recipe for the construction of $\psi(a)$, the arbitrarily given favorable argument w has the property that if x_0 is used neither at the $(w+1)$th step nor at any previous step, then there exists a number which is used at the $(w+1)$th step and is less than x_0. Among the infinitely many favorable arguments we can find x_0+1 of them, say, $w_1, \cdots, w_{x_0+1}$. It can not be that for every j $(1\leqq j\leqq x_0+1)$, x_0 is used neither at the (w_j+1)th step nor at any other step preceding to it. For if it were the case, there would be x_0+1 distinct numbers all less than x_0. This is impossible. Hence there must be a step, say, the $(p+1)$th step (where $p\leqq$some w_j) at which x_0 is used for the evaluation of $\psi(p)$. This number p is, of course, still a favorable argument of $\phi(g(a))$.

The number x_0 is, then, one among the $\pi(p)$ numbers $x_{p,1}, \cdots, x_{p,\pi}$ and is used to define one of the $\pi(p)$ partial recursive functions, say, $k_{p,j}(a)$. By definition, $\psi(p)$ is greater than $k_{p,j}(p)$ at least by 1. In fact, $k_{p,j}(a)$ is just the same function as $\phi(g(a))$. Thus $\psi(a)$ is not identically equal to $\phi(g(a))$. This completes the proof of the conjecture of Routledge.

References

1. N. A. Routledge, *Ordinal recursion*, Proc. Cambridge Philos. Soc. vol. 49 (1953) pp. 175–182.

2. R. Peter, *Rekursive Funktionen*, Budapest, Akademiai Kiado, 1951.

3. ———, *Konstruktion nichtrekursiver Funktionen*, Math. Ann. vol. 111 (1935) pp. 42–60.

4. S. C. Kleene, *General recursive functions of natural numbers*, Math. Ann. vol. 112 (1936) pp. 727–742.

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