AN ALGORITHM FOR DETERMINING WHETHER A GIVEN BINARY MATROID IS GRAPHIC

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1. **Introduction.** In a recent series of papers [1–4] on graphs and matroids I used definitions equivalent to the following. A *binary chain-group* N on a finite set M is a class of subsets of M forming a group under mod 2 addition. These subsets are the *chains* of N. A chain of N is *elementary* if it is non-null and has no other non-null chain of N as a subset. A *binary matroid* is the class of elementary chains of a binary chain-group.

As an example of a binary chain-group we may take the class of all *cuts* of a given finite graph G. A cut of G is determined by a partition of its set of vertices into two disjoint subsets U and V, and is defined as the set of all edges having one end in U and the other in V. I have called the corresponding binary matroid the *bond-matroid* of G. In the above-mentioned series of papers I obtained necessary and sufficient conditions for a given binary matroid to be *graphic*, that is representable as the bond-matroid of a graph.

On several occasions it has been pointed out to me that these results are of interest to electrical engineers,[1] but that a practical method for deciding whether or not a given binary matroid was graphic would be still more interesting. In what follows I present an algorithm which I hope will be of some use in this connection. This algorithm is described in §3 and the theorems needed to justify it are collected in §2.

2. **Theorems on binary matroids.** The *rank* of a binary chain-group N is the maximum number of chains linearly independent with respect to mod 2 addition. We denote it by $r(N)$.

The structure of N is uniquely determined by a *representative* matrix R. The columns of R correspond to the elements of M and the rows to the members of a set of $r(N)$ linearly independent chains of N. The elements of R are residues mod 2. The element in the ith row and jth column is 1 if the corresponding element of M belongs to the corresponding chain of N, and is 0 otherwise. It is clear that the chains of N correspond to the linear combinations of the rows of R, the total number of chains being $2^{r(N)}$.

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[1] The idea which led to this paper occurred to me during a conversation with Dr. S. Seshu. Another discussion of the problem has been given by Auslander and Trent [6].

It may happen that we can replace the elements of R by ordinary integers, the residue 0 by the integer 0 and the residue 1 by $+1$ or -1, in such a way that the determinant of each $r(N)\times r(N)$ submatrix of R takes the value 0, 1 or -1. Such matrices of integers and the associated "regular chain-groups" are discussed in [**1**] and [**3**]. If R has the property just stated we say that the binary matroid corresponding to R is *regular*.

Suppose we are given a representative matrix R of N. Then by elementary transformations of R, including possibly a permutation of the columns, we can obtain a new representative matrix R' of N in which the first $r(N)$ columns constitute a unit matrix. We call R' a *standard* representative matrix of N, or of the associated binary matroid.

THEOREM 1. *In a standard representative matrix R' of a binary chain-group N each row represents an elementary chain.*

PROOF. Suppose the chain K_i corresponding to the ith row is not elementary. Then it contains two non-null chains X and Y, where $X+Y=K_i$, one of which includes no element of M associated with any of the first $r(N)$ columns of R'. But this is impossible since X and Y must correspond to linear combinations of the rows of R'.

Let $\boldsymbol{M}$ be a binary matroid on a finite set M. We refer to the members of M as the *cells* of $\boldsymbol{M}$. Because of geometrical analogies pointed out in [**2**] we refer to the members of the class $\boldsymbol{M}$ as its *points*. (They are the "circuits" of the matroid in Hassler Whitney's terminology [**5**].)

A *separator* of $\boldsymbol{M}$ is a subset S of M such that no point of $\boldsymbol{M}$ meets both S and $M-S$. A separator is *elementary* if it is non-null and contains no other non-null separator. It is clear that the elementary separators of $\boldsymbol{M}$ are disjoint and that their union is M. In view of Theorem 1 they can be determined by inspection of any standard representative matrix of $\boldsymbol{M}$. We call $\boldsymbol{M}$ *connected* if it has no separator other than M and its null subset.

Let N be a binary chain-group on a set M, and let S be any subset of M. The class of all chains of N contained in S is a binary chain-group on S. We denote it by $N\times S$. The class of all intersections of S with chains of N is another binary chain-group on S which we denote by $N\cdot S$. If $\boldsymbol{M}$ is the matroid corresponding to N there are analogous definitions of matroids $\boldsymbol{M}\times S$ and $\boldsymbol{M}\cdot S$ on S. The points of $\boldsymbol{M}\times S$ are those points of $\boldsymbol{M}$ which are contained in S, and the points of $\boldsymbol{M}\cdot S$ are the minimal non-null intersections with S of points of $\boldsymbol{M}$ [**3**, §3]. $\boldsymbol{M}\times S$ and $\boldsymbol{M}\cdot S$ are the matroids corresponding to the chain-groups $N\times S$ and $N\cdot S$ respectively [**3**, (4.1) and (4.2)].

The following rules, proved in [4, §3] are useful in calculations with matroids.

(i) *If* $T \subseteq S \subseteq M$, *then*

$$(\mathbf{M} \times S) \times T = \mathbf{M} \times T,$$
$$(\mathbf{M} \cdot S) \cdot T = \mathbf{M} \cdot T,$$
$$(\mathbf{M} \cdot S) \times T = (\mathbf{M} \times (M - (S - T))) \cdot T,$$
$$(\mathbf{M} \times S) \cdot T = (\mathbf{M} \cdot (M - (S - T))) \times T.$$

(ii) *A subset S of M is a separator of* $\mathbf{M}$ *if and only if*

$$\mathbf{M} \cdot S = \mathbf{M} \times S.$$

Let Y be a point of a binary matroid $\mathbf{M}$ on a set M. We define the *bridges* of Y in $\mathbf{M}$ as the elementary separators of the matroid $\mathbf{M} \cdot (M - Y)$. To each such bridge B there corresponds a *Y-component* $\mathbf{M} \times (B \cup Y)$ of $\mathbf{M}$.

THEOREM 2. *If* $\mathbf{M}$ *is connected then each Y-component of* $\mathbf{M}$ *is connected.*

This theorem can be deduced from [4, (6.3)]. By the definition of a connected flat in [2, §1] the statement that $S \cup Y$ is a connected flat of $\mathbf{M}$ means that $\mathbf{M} \times (S \cup Y)$ is a connected matroid. Consider any Y-component $\mathbf{M} \times (B \cup Y)$ of $\mathbf{M}$, where B is a bridge of Y in $\mathbf{M}$. Putting $S = B$ in [4, (6.3)] we find that either $\mathbf{M} \times (B \cup Y)$ is connected or

$$(\mathbf{M} \cdot (M - Y)) \times B = \mathbf{M} \times B.$$

But

$$(\mathbf{M} \cdot (M - Y)) \times B = (\mathbf{M} \cdot (M - Y)) \cdot B, \qquad \text{by (ii)},$$
$$= \mathbf{M} \cdot B, \qquad \text{by (i)}.$$

Hence in the latter alternative $\mathbf{M}$ is not connected, by (ii), which is contrary to hypothesis.

For each bridge B of Y in $\mathbf{M}$ the matroid $(\mathbf{M} \times (B \cup Y)) \cdot Y$ is of interest. It may happen that its points are disjoint subsets $S_1, S_2, \cdots, S_k$ of Y whose union is Y. If so we say that B *partitions* Y, and that $\{S_1, S_2, \cdots, S_k\}$ is the *partition* of Y determined by B. Then each standard representative matrix of $(\mathbf{M} \times (B \cup Y)) \cdot Y$ has just one nonzero element in each column, and its rows correspond to the points S_i.

THEOREM 3. *If* $\mathbf{M}$ *is regular then each bridge of Y in* $\mathbf{M}$ *partitions Y* [4, (7.3)].

THEOREM 4. *Every graphic matroid is regular* [**4**, (4.1)].

Let B and B' be bridges of Y in $\boldsymbol{M}$ which partition Y, and let them determine partitions $\{S_1, \cdots, S_k\}$ and $\{T_1, \cdots, T_m\}$ of Y respectively. We call them *nonoverlapping* bridges if we can find S_i and T_j such that $Y = S_i \cup T_j$. In the remaining case B and B' *overlap*. We call Y an *even* point of $\boldsymbol{M}$ if it satisfies the following two conditions.

(a) Each bridge of Y in $\boldsymbol{M}$ partitions Y.

(b) The bridges of Y in $\boldsymbol{M}$ can be arranged in two disjoint classes so that no two members of the same class overlap.

In [**4**] even points were defined only for regular matroids. For them condition (a) can be omitted because of Theorem 3.

THEOREM 5. *In a graphic matroid every point is even* [**4**, (8.2)].

THEOREM 6. *Let $\boldsymbol{M}$ be the bond-matroid of a graph G. Let Y be a point of $\boldsymbol{M}$ having at most one bridge in $\boldsymbol{M}$. Then there is a vertex a of G such that Y is the set of all edges of G joining a to other vertices* [**4**, (4.13)].

THEOREM 7. *If $\boldsymbol{M}$ is graphic and $S \subseteq M$, then $\boldsymbol{M} \times S$ is graphic*, [**4**, (4.10)].

THEOREM 8. *Let Y be an even point of a connected binary matroid $\boldsymbol{M}$ such that every Y-component of $\boldsymbol{M}$ is graphic. Then $\boldsymbol{M}$ is graphic.*

The last theorem requires some comment. It derives from [**4**, (8.5)]. But the enunciation of [**4**, (8.5)] makes the additional postulates that $\boldsymbol{M}$ is regular and that Y has at least one bridge. The latter requirement is not important. For if Y has no bridges it is the only point of $\boldsymbol{M}$, and then $\boldsymbol{M}$ is trivially graphic.

The postulate of regularity in [**4**, (8.5)] is not necessary. It is used only in the appeal to [**4**, (7.4)] and in the proof that $\boldsymbol{M} \times (U_i \cup Y)$ is regular. But the argument of [**4**, (7.4)] can be applied to any binary matroid $\boldsymbol{M}$. It shows that if B partitions Y in $\boldsymbol{M}$ it determines the same partition of Y in $\boldsymbol{M} \times S$, (where $B \cup Y \subseteq S$). Moreover with the above enunciation it is unnecessary to prove $\boldsymbol{M} \times (U_i \cup Y)$ regular as a step towards proving it graphic.

3. The algorithm.

Suppose we are given a connected binary matroid on a set M. We can determine whether or not it is graphic by the following procedure.

First we construct a standard representative matrix R'. If no column of R' has more than two nonzero elements we form the mod 2 sum of the rows of R', adjoin it to R' as an extra row, and so obtain the incidence matrix of a graph whose bond-matroid is $\boldsymbol{M}$. In the remaining case we may suppose, without loss of generality that the last column of R' has nonzero elements in the first, second and third rows.

The first row corresponds to a point Y of $\boldsymbol{M}$, by Theorem 1. Striking out from R' the first row and all columns having nonzero elements in the first row we obtain a standard representative matrix R'' of $\boldsymbol{M}\cdot(M-Y)$. From R'' we obtain the elementary separators of $\boldsymbol{M}\cdot(M-Y)$, that is the bridges $B_1, \cdots, B_m$ of Y in $\boldsymbol{M}$.

We may find that Y has only one bridge in $\boldsymbol{M}$. If so we repeat the process with the point of $\boldsymbol{M}$ corresponding to the second row. If this point has only one bridge we proceed to the third row. If this too corresponds to a point with only one bridge we may assert that $\boldsymbol{M}$ is not graphic. For suppose $\boldsymbol{M}$ is the bond-matroid of a graph G. Then by Theorem 6 the last column of R' corresponds to an edge of G having three distinct ends.

In the remaining case we may suppose without loss of generality that Y has at least two bridges in $\boldsymbol{M}$.

For each bridge B_i we determine the corresponding Y-component $\boldsymbol{M}\times(B_i\cup Y)$. A standard representative matrix of this can be obtained as follows. We take those rows of R' which are extensions of the rows of R'' representing chains in B_i, adjoin the first row of R', and then suppress all the zero columns of the resulting matrix. We next construct a standard representative matrix of $(\boldsymbol{M}\times(B_i\cup Y))\cdot Y$ to see if B_i partitions Y. If it does not we can assert that $\boldsymbol{M}$ is nonregular, by Theorem 3, and therefore nongraphic, by Theorem 4.

In the remaining case each bridge B_i partitions Y. We examine the partitions to see which bridges overlap, and thus determine whether or not Y is even. If it is not we can assert that $\boldsymbol{M}$ is not graphic, by Theorem 5.

If Y is even we have simplified the problem. For, by Theorems 7 and 8, $\boldsymbol{M}$ is graphic if and only if its Y-components are all graphic. But each of these Y-components is connected, by Theorem 2, and has lower rank than $\boldsymbol{M}$. We repeat the above procedure for the Y-components of $\boldsymbol{M}$, noting that we already have standard representative matrices for them, and continue in this way until the process terminates.

To conclude we observe that if we can obtain graphs corresponding to the Y-components of a graphic matroid $\boldsymbol{M}$ we can construct from them a graph corresponding to $\boldsymbol{M}$. The necessary constructions are described in the course of the proofs of Theorems (8.4) and (8.5) of [4].

The algorithm can be used to decide whether a given graph is planar. For a planar graph is simply a graph whose circuit-matroid, the dual of its bond-matroid, is graphic.

4. **An example.** Consider the connected binary matroid $\boldsymbol{M}$ defined by the following standard representative matrix R_1.

$$\begin{array}{ccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15
\end{array}$$
$$\begin{Bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1
\end{Bmatrix}$$

There are several columns with three or more 1's. Let us work with Column 13, without bothering to put it in the last place. The first row with a 1 in this column is the fifth. Let this correspond to a point Y of $\boldsymbol{M}$.

Striking out row 5 and every column having a 1 in this row we obtain the following standard representative matrix R_2 of $\boldsymbol{M}\cdot(M-Y)$.

$$\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 6 & 7 & 14 & 15
\end{array}$$
$$\begin{Bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{Bmatrix}$$

We observe that the elementary separators of $\boldsymbol{M}\cdot(M-Y)$, the bridges of Y in $\boldsymbol{M}$, are $B_1=\{6\}$, $B_2=\{1, 7, 15\}$ and $B_3=\{2, 3, 4, 14\}$, where cells of $\boldsymbol{M}$ are represented by the numbers of the associated columns in R_1. The general rule for constructing an elementary separator is to take an arbitrary row of R_2, then every row having a 1 in the same column as a 1 of the first row taken, then every row having a 1 in the same column as a row already chosen, and so on. The elementary separator is determined by the 1's of the resulting submatrix.

The Y-components corresponding to the bridges B_i $(i=1, 2, 3)$

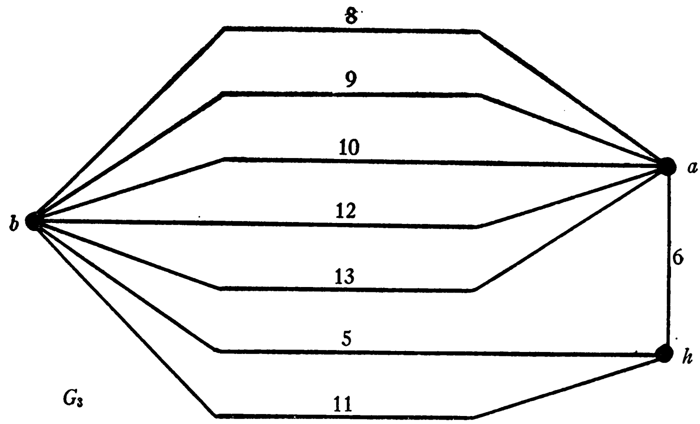
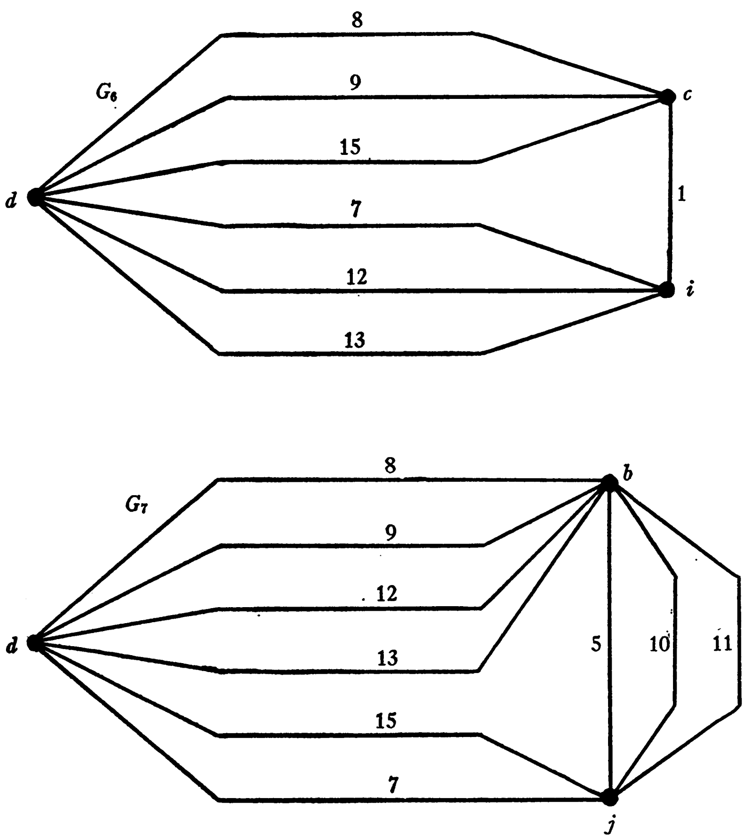

8
9
10
12
13
5
11
a
b
6
h
G3

FIG. 1


8
9
15
7
12
13
c
d
1
i
G6
8
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7
b
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5
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11
j
G7

FIG. 2

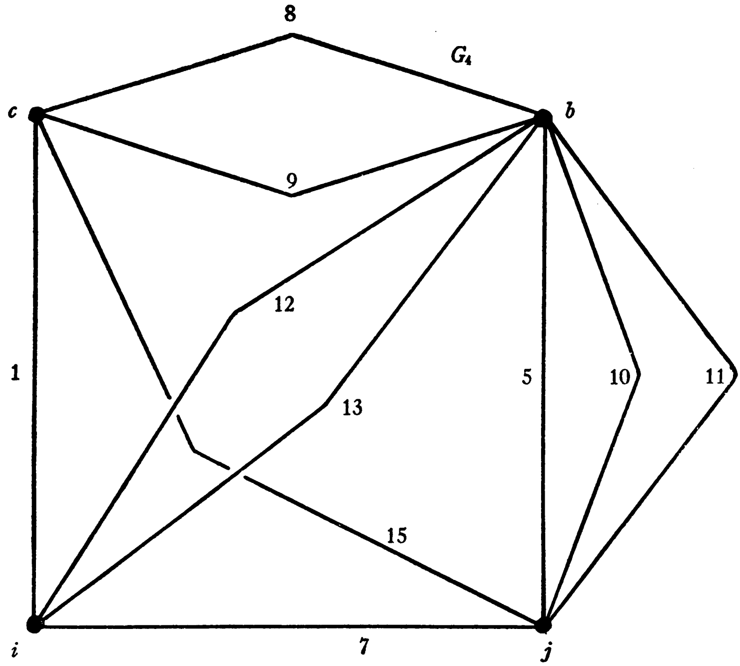

FIG. 3

are represented, in order, by the following three submatrices of R_1. In each submatrix the last row represents Y.

$$
\begin{array}{ccccccccccccccccc}
 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15
\end{array}
$$

$$
R_3 = \left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{matrix}\right\}
$$

$$
R_4 = \left\{\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{matrix}\right\}
$$

$$
R_5 = \left\{\begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{matrix}\right\}
$$

If zero columns are ignored these are all standard representative matrices, to within a permutation of the rows. Let the corresponding matroids be $\boldsymbol{M}_3$, $\boldsymbol{M}_4$ and $\boldsymbol{M}_5$ respectively.

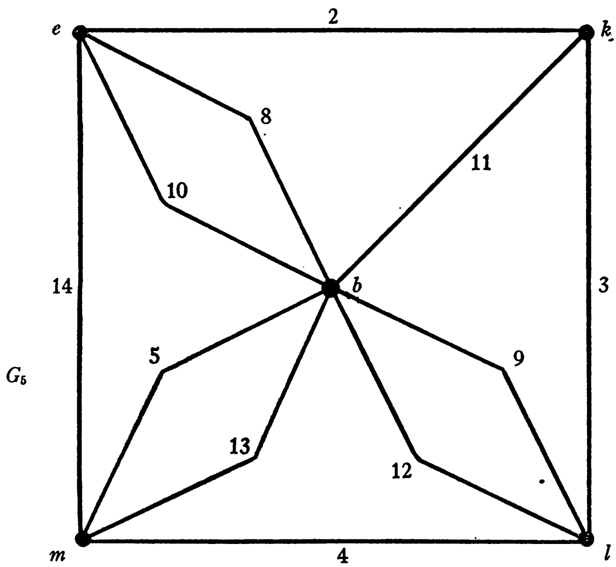

FIG. 4

Our next question is: do B_1, B_2 and B_3 partition Y? To answer it in the case of B_3 we strike out all the columns of R_5 having a 0 in the last row, thus obtaining the following representative matrix of $M_5 \cdot Y = (M \times (B_3 \cup Y)) \cdot Y$.

$$\begin{array}{ccccccc} 5 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}$$
$$\begin{Bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{Bmatrix}$$

This can be reduced to standard form (to within a permutation of columns) by adding the first row to the others, then the new second row to the third and fourth, and then the new third row to the fourth.

$$\begin{array}{ccccccc} 5 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}$$
$$\begin{Bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{Bmatrix}$$

The standard representative matrix has only one 1 in each column. We may therefore assert that B_3 partitions Y. The corresponding partition, P_3 say, is $\{\{5, 13\}, \{8, 10\}, \{9, 12\}, \{11\}\}$.

Similarly we find that B_1 and B_2 determine partitions $P_1 = \{\{8, 9, 10, 12, 13\}, \{5, 11\}\}$ and $P_2 = \{\{5, 10, 11\}, \{8, 9\}, \{12, 13\}\}$ of Y respectively.

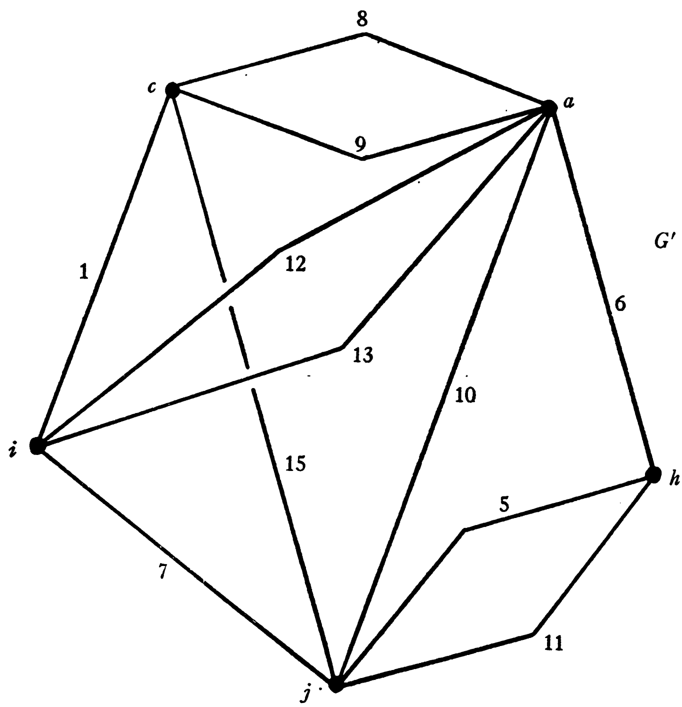

FIG. 5

If the three bridges had not all partitioned Y,—if for example the standard representative matrix of $M_4 \cdot Y$ had had two 1's in one of its columns,—the algorithm would have terminated here and we would have written off M as nongraphic. As things are we must go on to investigate whether Y is even. This is not difficult. B_1 and B_2 do not overlap since the unions of the member $\{8, 9, 10, 12, 13\}$ of P_1 and the member $\{5, 10, 11\}$ of P_2 is the whole of Y. Hence Y is even; its bridges can be arranged in two disjoint classes $U = \{B_1, B_2\}$ and $V = \{B_3\}$ so that no two members of the same class overlap. If we had found that Y was not even we would thereby have proved M nongraphic. As it is we have completed the first stage of the algorithm and we can assert that $\boldsymbol{M}$ is graphic if and only if $\boldsymbol{M}_3$, $\boldsymbol{M}_4$ and $\boldsymbol{M}_5$ are all graphic. It remains to apply the algorithm to R_3, R_4 and R_5.

$\boldsymbol{M}_3$ is graphic because R_3 has at most two 1's in each column.

R_4 has three 1's in the eighth column. The points of $\boldsymbol{M}_4$ corresponding to the first and third rows are found to have one bridge each. (We ignore zero columns.) In the case of the third row this is ensured

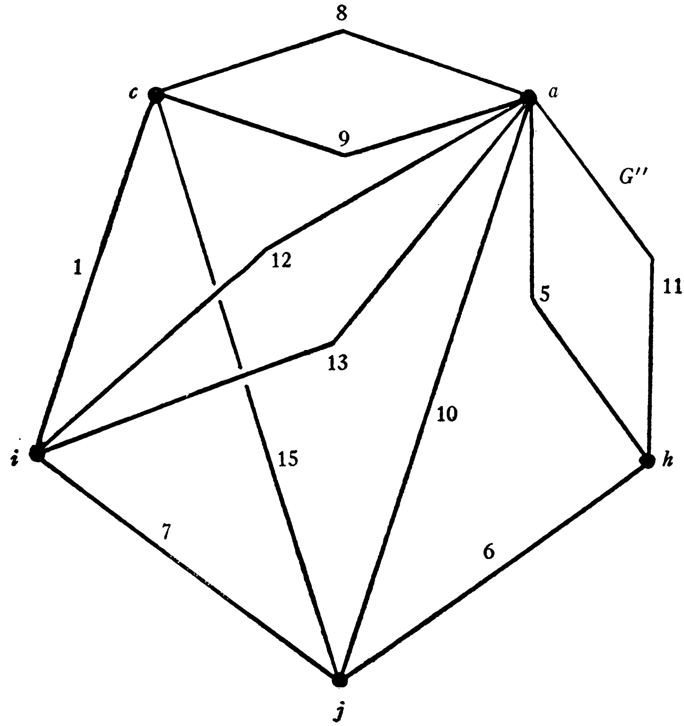

FIG. 6

by the construction of R_4. If the same result were found for the second row then $\boldsymbol{M}_4$ and $\boldsymbol{M}$ would be proved nongraphic. Actually the point of $\boldsymbol{M}_4$ corresponding to the second row,—Y' say,—has two bridges. The submatrices of R_4 corresponding to the Y'-components are R_6, consisting of the first and second rows, and R_7, consisting of the second and third. Since R_6 and R_7 have only two rows each they represent graphic matroids. Hence both bridges partition Y', by Theorems 3 and 4. Since Y' has only two bridges it is even. The algorithm thus shows $\boldsymbol{M}_4$ to be graphic.

$\boldsymbol{M}_5$ is also found to be graphic. We do not give the analysis in detail. We remark however that it can be carried out in two steps. The first replaces R_5 by R_8, consisting of the first two rows, and R_9, consisting of the last three. The second replaces R_9 by R_{10}, consisting of the second and third rows of R_5, and R_{11}, consisting of the third and fourth.

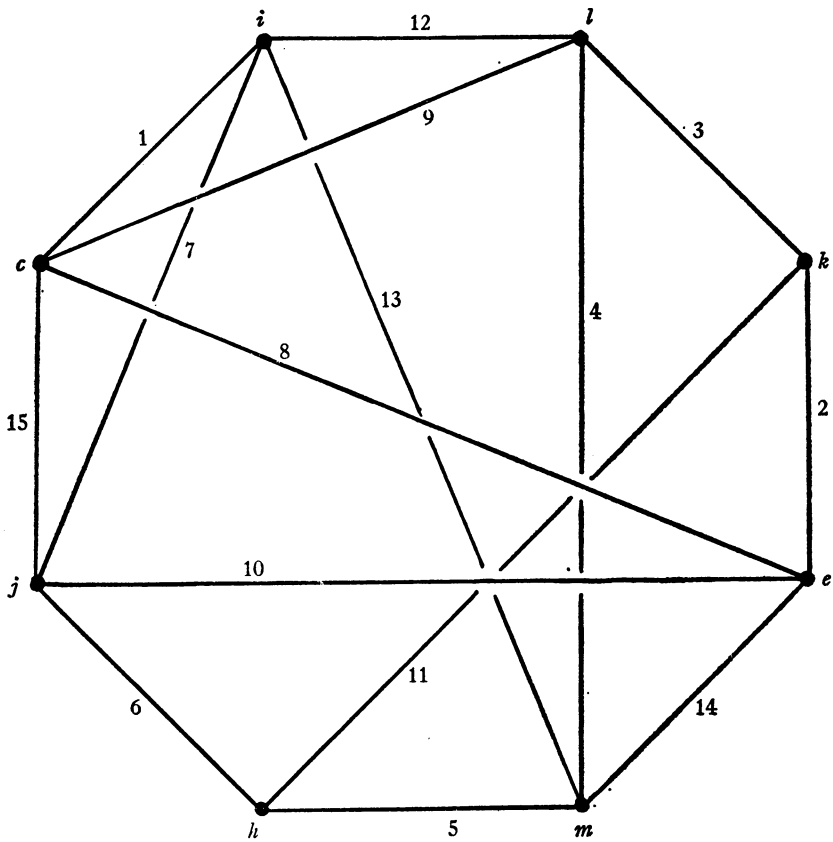

FIG. 7

We conclude that M is graphic.

The construction of the corresponding graph may be of interest. Let G_i denote a graph whose bond-matroid is represented by R_i. The construction of G_3 from R_3 is trivial, and this graph is represented in Figure 1.

G_6 and G_7 are represented in Figure 2. To construct G_4 we combine them, with elimination of the vertex d (Figure 3). G_9 can be constructed similarly from G_{10} and G_{11}. It can then be combined with G_8, with elimination of a vertex, to produce G_5 (Figure 4).

The combination of the three graphs G_3, G_4 and G_5 is more complicated. The general rule is as follows: first combine the graphs corresponding to U, then combine those corresponding to V, and then unite the two resulting graphs. So in the case under consideration we combine G_3 and G_4, with elimination of b, to produce a graph G' (Figure 5).

Direct combination of G' and G_5 is impossible. We appeal however to the following well-known rule. Let H be a part of a graph G joined to the rest only at two vertices x and y. Let L be formed from G by reversing H. This means that the edges of H incident with $x(y)$ in G are those incident with $y(x)$ in L, and that all the other incidence relations are unchanged. Then the cuts of G and L are the same and therefore their bond-matroids are identical. Applying this rule to G' we obtain the graph G'' of Figure 6. Its bond-matroid is represented by the submatrix of R_1 which is the union of R_3 and R_4. We can now combine G'' and G_5, with elimination of b in G_5 and a in G'', to produce the graph G_1 of Figure 7 whose bond-matroid is $\boldsymbol{M}$.

The existence of a reversal in G' making the edges of Y incident with a common vertex is not fortuitous. Such a reversal, or sequence of reversals, can always be found when Y has no overlapping bridges [**4**, (8.4)].

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