POWER SERIES WITH GAPS[1]

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If every qth term in a power series $f(z)=a_0+a_1z+\cdots$ with finite radius of convergence is missing then $f(z)$ has at least two singularities on its circle of convergence (Mandelbrojt [1]). I proved more generally [2, Theorem 1]:

If $a_m=0$ for all $m\equiv\tau_1,\cdots,\tau_k \pmod q$, $k\leqq 3$, where $\tau_1,\cdots,\tau_k$ for every divisor q' of q belong to $\max(k, q')$ *different residue classes, then $f(z)$ has at least $k+1$ singularities on its circle of convergence.*

I wish to show here that this statement *is not true for $k=4$ and $k\geqq 6$*. The case $k=5$ remains open. It would be of interest to determine the minimum number ν of singularities for given k, as well as for given k and q. For prime q it is known [3] that $\nu\geqq k+1$.

The pertinent counterexample is

$$f(z)=\frac{1-z^{\alpha}}{1-z^{\alpha\beta}}=1-z^{\alpha}+z^{\alpha\beta}-z^{\alpha+\alpha\beta}+\cdots,$$

with $q=\alpha\beta$ and $\alpha\beta-\alpha$ singularities on the unit circle. Let $\tau_1,\cdots,\tau_k$ be the $q-\gamma$ numbers $0,\cdots,q-1$ after deletion of $0,\cdots,\gamma-2$ and α, where $2\leqq\gamma\leqq\alpha$. If $\beta\geqq 3$ then, for every proper divisor q' of q, all residue classes are represented by $q-q',\cdots,q-1$. However, $\alpha\beta-\alpha\leqq k$.

For given k, the choice of α (not exceeding $k/2$ and not dividing $k+1$) determines β and γ uniquely. Set $\alpha=[k/2]$, $\beta=3$, $\gamma=k/2$ or $(k-3)/2$; then $\gamma\geqq 2$ excludes only $k=1, 2, 3, 5$.

References

1. S. Mandelbrojt, *Sur les séries de Taylor qui présentent des lacunes*, Ann. Sci. Ecole Norm. Sup. (3) vol. 40 (1923) pp. 413–462.

2. T. S. Motzkin, *Bemerkung über Singularitäten gewisser mit Lücken behafteter Potenzreihen*, Math. Ann. vol. 109 (1933) pp. 95–100; in Theorem 1, Condition 1 is to be taken in its stronger meaning: "in" classes does not imply exhaustion of those classes.

3. A. Ostrowski, *Mathematische Miszellen.* VII: *Über Singularitäten gewisser mit Lücken behafteten Potenzreihen*, Jber. Deutsch. Math. Verein. vol. 35 (1926) pp. 269–280.

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