FIXED-POINT THEOREMS FOR ARCWISE CONNECTED CONTINUA

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L. E. Ward, Jr., recently proved in [4] a fixed-point theorem for certain arcwise connected spaces that generalizes a theorem of mine—Theorem 2, below—and Borsuk's theorem [1] that an arcwise connected hereditarily unicoherent metric curve has the fixed-point property. His argument provides a proof of my result, but not of Borsuk's. That Borsuk's class of continua is contained in his follows from Borsuk's result only.

In this note I give a new sufficient condition for the fixed-point property that implies Borsuk's result, and that follows from my theorem and so from Ward's. I also give an example of an arcwise connected continuum that contains no simple closed curve but that does not have the fixed-point property, and prove a fixed-point theorem for a quite special class of contractible continua.

**Theorem 1.** Let M be an arcwise connected compact Hausdorff space that does not have the fixed-point property. Then M contains either (1) a continuum N₁ for which there is a map f: N₁ → S¹ which is onto and such that no closed proper subset of N₁ is mapped by f onto S¹, and which is such that at most one point-inverse is nondegenerate, that one being connected; or (2) a continuum N₂ that contains a subset R that is the one-to-one continuous image of a half-open interval and that is dense in N₂, but that has no interior relative to N₂; or (3) a continuum N₃ that is the union of a set R that is the continuous one-to-one image of a half-open interval, and a continuum B, and for which there is a map f: N₃ → K, K being the union of the circles x² + y² = (2/n)y, n = 1, 2, 3, ..., such that f is one-to-one on N₃ − B, such that f(B) = (0, 0), and such that no closed proper subset of N₃ is mapped by f onto K.

We will see that this is a consequence of an earlier fixed-point theorem of the author's, proved in [5, p. 493]:

**Theorem 2.** Let M be an arcwise connected Hausdorff space which is such that every monotone increasing sequence of arcs is contained in an arc. Then M has the fixed-point property.

Note that compactness is not required in Theorem 2.

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Proof of Theorem 1. The proof is a straightforward analysis of the possible ways the hypothesis of Theorem 2 can fail in a compact space. Let $A_1, A_2, A_3, \ldots$ be a monotone increasing sequence of arcs that is not contained in an arc. Let $x$ be any non-end point of $A_1$. Then, for each $n$, $x$ divides $A_n$ into two arcs $A_n'$ and $A_n''$, the primes being chosen so that, for each $n$, $A_n'$ is contained in $A_{n+1}'$. If $M$ contains a simple closed curve, then that is a continuum of type 1. Suppose that $M$ contains no simple closed curve. Then at least one of the two monotone increasing sequences $\{A_n'\}, \{A_n''\}$ does not lie in an arc. Hence there is no loss in assuming that $\{A_n\}$ itself is a sequence of arcs all having a common end point, $a$. There is also no loss in assuming that $A_{n+1}'-A_n$ is never empty. Let $B$ denote the set $\lim \sup C(A_{n+1}'-A_n)$. An argument of a familiar type shows that $B$ is connected. For if $B$ were not, there exist two disjoint open sets $U$ and $V$, covering $B$, and each intersecting $B$. From some integer $k$ on, each set $C(A_{n+1}'-A_n)$ lies either in $U$ or in $V$. But $C(A_{n+1}'-A_n)$ and $C(A_{n+2}'-A_{n+1}')$ intersect. Induction shows that for $n>k$, either all the sets $C(A_{n+1}'-A_n)$ lie in $U$, or all lie in $V$. This gives a contradiction.

The set $\bigcup_n A_n = R$ is the one-to-one continuous image of a half-open interval. There are three possible relations between $R$ and $B$: (1) The sets $R$ and $B$ are disjoint. Then there is an arc $xy$ from some point $x$ of $B$ to some point $y$ of $R$, such that $x = xy \cap B$ and $y = xy \cap R$. The point $y$ separates $R$ into two connected sets, $R'$ and $R''$, where $R'' \cup y$ is an arc from $a$ to $y$, and $R' \cup y$ is again the one-to-one continuous image of a half-open interval. (In this case, it is actually a homeomorph of such an interval.) Let $N_1 = xy \cup R' \cup B$. The collection consisting of the set $B$ and of the individual points of $N_1 - B$ is upper semi-continuous and defines a map $f: N_1 \to S^1$ satisfying the conditions of part (1) of the conclusion of the theorem. (2) The sets $R$ and $B$ intersect, but some arc $au$ of $R$ contains $R \cap B$. (It may actually happen that $R \cap B$ consists of just two points.) Let $R'$ be the set $R - au$, and let $N_1 = R' \cup B$. Then in the same way as above, we have the desired map $f: N_1 \to S^1$. (3) There is an integer $k$ such that $\bigcup_{n=k}^{\infty} (A_{n+1}'-A_n) = R'$ is contained in $B$. Then $N_2 = B$ is the desired continuum of the second part of the conclusion, with $R' = R$. (4) No arc of $R$ contains $R \cap B$, and also there is no integer $k$ such that $\bigcup_{n=k}^{\infty} (A_{n+1}'-A_n)$ is contained in $B$. In this case, $UA_n - B$ is the union of a countable number of disjoint open intervals, $I_1, I_2, I_3, \ldots$. Let $N_3 = B \cup \bigcup I_n$. The upper-semicontinuous collection consisting of $B$ and of the individual points of the intervals $\{I_n\}$ defines a map of $N_3$ onto a continuum of the third type of the conclusion of Theorem 1, satisfying the desired conditions.
From Theorem 1, we get an easy proof of Borsuk's theorem.

**Theorem 3.** If \( M \) is an arcwise connected, hereditarily unicoherent metric [or Hausdorff] curve, then \( M \) has the fixed-point property.

**Proof.** Note that a continuum of either of the first or third types described in Theorem 1 is not unicoherent, so that \( M \) contains neither of these. Next, a hereditarily unicoherent arcwise connected continuum \( M \) contains no indecomposable subcontinuum. For suppose that \( S \) is such an indecomposable subcontinuum of \( M \). There is an arc \( A \) in \( M \) whose end points lie in different composants of \( M \). Then \( A \) is not a subset of \( M \), and \( A \cup S \) is not unicoherent. Theorem 3 follows then from the next result, which seems to have escaped publication, and which shows that \( M \) can contain no continuum of the second type.

**Theorem 4.** If a hereditarily unicoherent continuum \( S \) contains a dense subset \( R \) that is the one-to-one continuous image of a half-open interval, but that contains no interior points, then \( S \) is indecomposable.

**Proof.** Suppose that \( S \) is the union of two proper subcontinua, \( A \) and \( B \). Each has an interior, \( \text{Int} \, A = S - B \) and \( \text{Int} \, B = S - A \), relative to \( S \). We may order the points of \( R \) by their order in the half-open interval, the image of the end point being the first point of \( R \). Let \( a_1 \) be a point of \( R \cap \text{Int} \, A \), \( b \) be a point of \( R \cap \text{Int} \, B \) that follows \( a_1 \) in \( R \), and \( a_2 \) be a point of \( R \cap \text{Int} \, A \) that follows \( b \) in \( R \). Then if \( a_1a_2 \) denotes the arc of \( R \) from \( a_1 \) to \( a_2 \), \( A \cup a_1a_2 \) is not unicoherent.

The join, in the sense of combinatorial topology, of a Cantor set and a point contains a subset \( R \) that is the continuous image of a half-open interval, that is dense in the join, and that has no interior, showing that the one-to-one property is required.

Theorem 1 does not imply Theorem 2. In fact, for each integer \( n > 1 \), there is an arcwise connected, contractible and metric continuum containing no subcontinuum of any of the three types of Theorem 1. Let \( X \) be a continuum of dimension \( n - 1 \) that contains no arc; for example, the product of \( n - 1 \) pseudo-arcs [3]. Let \( M \) be the join of \( X \) and a point \( p \). If \( S \) is a continuum in \( M \) of one of the three types, \( S - p \) cannot lie in one interval of the join, and the projection of \( M - p \) onto \( X \) will map some arc of \( S - p \) onto a nondegenerate continuum in \( X \). However, Borsuk's hypothesis cannot hold in \( X \), since a continuum of dimension greater than one cannot be hereditarily unicoherent. We can modify the example slightly, by replacing \( M \) by two such joins, having in common only one point, on the base of each, and show that for each integer \( n > 1 \), there is an arcwise connected and noncontractible metric continuum containing no subcontinuum of any of the three types of Theorem 1.
Kinoshita gave an example \[2\] of a contractible continuum that has no fixed point. Since it contains a 2-cell, it contains continua of all the types of Theorem 1. That result, however, does imply one fixed-point theorem for contractible continua.

**Theorem 5.** If \(M\) is a contractible Hausdorff continuum such that each two points are the end points of only one arc, then \(M\) has the fixed-point property.

**Proof.** Suppose that \(M\) contains a continuum \(N\) satisfying condition (1) of Theorem 1. The uniqueness of arcs shows that \(N\) cannot be a simple closed curve, so that one point-inverse under the mapping \(f\) of that condition is a nondegenerate continuum, \(B\). The proof of part (1) of Theorem 1 shows that we can assume that \(N\) is the union of \(B\) and the continuous one-to-one image \(R\) of a half-open interval, \(R \cap B\) consisting of the image of the end-point of that interval. Let \(c: M \times I \to M\) be a contraction, satisfying \(c(x, 1) = p\). By uniform continuity of \(c\), for each positive number \(\epsilon\), there is a positive number \(\delta\) such that if \(d(x, y) < \delta\), then for all \(t\) in \(I\), \(d[c(x, t), c(y, t)] < \epsilon\).

Let \(y\) be a point of \(B\) not in \(R\) and not \(p\). The set \(c(y \times I) \cap R\) may be empty, but if not, it is connected. For if \(c(y \times I) \cap R = H \cup K\), separated, then there exists an arc \(A_1\) in \(R\) from a point \(h\) in \(H\) to a point \(k\) in \(K\) and there is an arc \(A_2\) in \(c(y \times I)\) from \(h\) to \(K\), and \(A_1 \cup A_2\) contains a simple closed curve. If \(e\) denotes the end point of \(R\), which is in \(B\), it is conceivable that \(e\) is not in \(c(y \times I)\). It is not possible, however, that for some point \(x\) in \(R\), \(c(y \times I)\) contains the set \(R_x\) consisting of all the points \(z\) in \(R\) such that \(x\) is on the arc \(ez\) of \(R\). For suppose that this occurred. Then \(c(y \times I)\) contains \(B\). Let \(U\) be a relatively open connected subset of the Peano continuum \(c(y \times I)\) that contains \(e\) (which is in \(N\)), but does not contain \(x\). Let \(x'\) be a point of \(R_x \cap U\). There is an arc \(x'e\) in \(U\), and in \(R\) there are arcs \(ex, xx'\). The union \(x'e \cup ex \cup xx'\) contains a simple closed curve, which is impossible. We can thus conclude that \(R - c(y \times I)\) contains a set \(R_x - x\), for some \(x\) in \(R\); \(x\) will be in \(c(y \times I)\). If \(z\) is a point of \(R_x\), then \(c(z \times I)\) contains the arc \(xz\) of \(R\); otherwise \(xz \cup c(z \times I) \cup c(y \times I)\) contains a simple closed curve.

Now let \(\epsilon_1, \epsilon_2, \epsilon_3, \ldots\) be a sequence of positive numbers approaching 0. For each \(\epsilon_n\), let \(\delta_n\) be the corresponding number \(\delta\) defined in the last sentence of the first paragraph of this proof, and let \(x_n\) be a point of \(R_x\) within \(\delta_n\) of \(y\). Then \(d[c(x_n, t), c(y_0, t)] < \epsilon_n\) for all \(t\) in \(I\). Let \(z\) be a fixed point of \(R_x\); there is no loss in supposing that \(z\) is in each arc \(xx_n\) in \(R\). Then by our last paragraph, \(z\) is in each set \(c(x_n \times I)\). For each \(\epsilon_n\), then, \(d(z, c(y \times I)) < \epsilon_n\), so that \(z\) belongs to the
set $c(y \times I)$. But this is a contradiction.

Modifications of this argument dispose of each of the other two cases.

Either from Theorem 5 or, quicker, from Borsuk's theorem, it follows that a one-dimensional contractible continuum $C$ has the fixed-point property, since every subcontinuum is homologically acyclic, so that $C$ contains no simple closed curve.

Let $C_1$ be a continuum in the lower half $xy$-plane joining the point $(2, 0, 0)$ to the interval $[-3, -1]$ of the $x$-axis, $C_1$ being homeomorphic to the closure of the graph of $y = \sin 1/x$, $0 < x \leq \pi$, with the interval $[-3, -1]$ corresponding to the limiting interval of the graph. Let $C_2$ be the image of $C_1$ under the rotation of the $xy$-plane about the origin through an angle of $\pi$. Let $L_1$ and $L_2$ be straight-line intervals joining $(2, 0, 0)$ and $(-2, 0, 0)$ to $(0, 0, 1)$. Let $R$ be a set homeomorphic to a half-open interval that (1) has only $(0, 0, 1)$ in common with $C_1 \cup C_2 \cup L_1 \cup L_2$ and (2) "spirals down" to $C_1 \cup C_2$ in such a way that (a) there is a sequence of arcs $X_1, X_2, X_3, \ldots$ filling up $R$ such that $X_i \cap X_j$ is empty for $j \neq i + 1, i - 1$, and is an end point of each for $j = i + 1, i - 1$, and (b) $C_1 = \lim X_{2j}$ and $C_2 = \lim X_{2j+1}$. Let $M = C_1 \cup C_2 \cup L_1 \cup L_2 \cup R$. Then $M$ is arcwise connected by unique arcs, and is compact. We define a continuous map $f: M \to M$ that has no fixed point. Let $f_1: M \to M$ be a map that on $C_1 \cup C_2 \cup L_1 \cup L_2$ is the rotation of $E^3$ about the $Z$-axis through an angle of $\pi$, and that on $R$ is the identity; $f_1$ is not continuous. Let $f_2: M \to M$ be a map that is a homeomorphism on $R$ and maps each arc $X_n$ onto $X_{n+1}$ that is the identity on $C_1 \cup C_2$, and that maps each set $L_{ij}, j = 1, 2$, homeomorphically onto $L_j \cup X_1$, the points $(2, 0, 0)$ and $(-2, 0, 0)$ being kept fixed; $f_2$ is not continuous either. The composition $f = f_2 f_1$, however, is continuous, and no point is left fixed.

I have no such example in the plane, nor do I have a continuum $M$ that does not have the fixed-point property for homeomorphisms.

**Bibliography**


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