DETERMINANTS WHOSE ELEMENTS HAVE EQUAL NORM[1]

T. S. MOTZKIN

The following theorem on vanishing 3 by 3 determinants I proved previously [1] in the case where F is the field R of reals; the three proofs given there rely on special properties of the complex domain.

Let F be a field of arbitrary characteristic, F^* an extension of degree 2 of F. The conjugate of an element α of F^* is denoted by $\bar{\alpha}$; the norm of α is $\alpha\bar{\alpha}$.

THEOREM 1. *A vanishing 3 by 3 determinant of elements of F^* of equal norm has two proportional rows or columns.*

Let $\nu \neq 0$, where ν is the common norm. Then if the terms of the determinant are A, B, C, A', B', C', with

$$A + B + C = A' + B' + C', \tag{1}$$

the product of the 9 elements is

$$ABC = A'B'C'. \tag{2}$$

From (1) we have $\bar{A}+\bar{B}+\bar{C}=\bar{A}'+\bar{B}'+\bar{C}'$; multiplying by (2) and dividing by ν^3 we obtain

$$AB + AC + BC = A'B' + A'C' + B'C'. \tag{3}$$

Hence A, B, C equal A', B', C' in some order. Each of the 6 orders leads immediately to the proportionality of two rows or columns.

The above theorem, in its specialization to minors of Vandermonde determinants composed of qth roots of unity in R^*, was used in [1] for the proof of a theorem on power series without terms whose subscript belongs to one of 3 residue classes modulo an arbitrary integer q. In [2] I showed that the corresponding theorem for 4 residue classes is false for $q=6$. This suggests that

There exist vanishing 4 by 4 minors of the form $|\epsilon^{a_j b_k}|$, $\epsilon^q=1$, ϵ *in* R^*, *without proportional rows or columns.*

Indeed, examining the counterexample in [2] in the light of the proof in [1] we obtain the determinant

---

Presented to the Society, April 23, 1960; received by the editors August 10, 1959.

[1] Sponsored, in part, by the Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the United States Government.

$$D = \begin{vmatrix} \epsilon & \epsilon^2 & -\epsilon & -\epsilon^2 \\ -1 & 1 & 1 & -1 \\ -\epsilon & \epsilon^2 & -\epsilon & \epsilon^2 \\ -\epsilon^2 & -\epsilon & \epsilon^2 & \epsilon \end{vmatrix} = \left| \epsilon^{a_j b_k} \right| = 0,$$

$$a_j = 1, 3, 4, 5, \ b_k = 1, 2, 4, 5,$$

where ϵ is a primitive 6th root of unity, and D has no proportional rows or columns. It is known [**3**] that such determinants do not exist for any degree if q is prime.

However, D has exactly 4 vanishing 3 by 3 minors, viz. those in rows 1, 2, 4; these rows are therefore dependent. The existence of vanishing 4 by 4 determinants (even other than Vandermonde minors) composed of roots of unity in R^* and without dependent rows or columns is still in question. But if the elements are only required to have equal norm we can show:

There exist vanishing 4 *by* 4 *determinants whose elements have norm* 1 *in* R^* *and none of whose minors is* 0.

Replacing the two last elements of D by x and y we obtain an equation $ax+by+c=0$ that is also fulfilled by $x=1$, $y=-1$. Hence when x traces the unit circle U, y traces a circle V meeting U in two points. If the 14 first elements of D are now slightly moved on U, so that all 2 by 2 minors involving them only become nonzero, then a, b, c and V move somewhat. Since V continues to meet U, values x and y of norm 1 can still be supplied; but now no 3 by 3 minor can vanish, as this would imply the vanishing of a 2 by 2 minor not involving x and y. Nor can the product of all 2 by 2 minors involving x or y or both vanish for all small changes as indicated, since it would have to vanish identically; none of these minors, however, vanishes both for $x=-\epsilon$, $y=-\epsilon^2$ and for $x=1$, $y=-1$.

As in [**1**] Theorem 1 implies

THEOREM 2. *A matrix of rank* 2 *of elements of* F^* *of equal norm consists either of all rows or of all columns of two matrices of rank* 1.

Theorem 1 does not hold for general extension fields. We have, e.g.:

There exist vanishing 3 *by* 3 *determinants without proportional rows or columns, composed of elements of equal norm of a cubic extension* F^* *of the field* $F=R_0(\epsilon)$ *where* $\epsilon^2+\epsilon=-1$ *and* R_0 *is the field of rationals.*

The determinant

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \epsilon & x \\ 1 & y & \epsilon^2 \end{vmatrix} = 2 + x + y - xy$$

vanishes if and only if

$$(x-1)(y-1)=3, \tag{4}$$

but has no proportional rows or columns unless (x, y) or $(y, x)=(\epsilon, \epsilon^2)$. If $f(x)=x^3+cx^2+(c-3)x-1$ is irreducible over F, i.e., if $f(x)\neq 0$ for $x=\pm 1, \pm\epsilon, \pm\epsilon^2$, then $F^*=F(x)$ with $f(x)=0$ is a cubic extension in which the norm of x is $-f(0)=1$. Since $(x-1)^2+(c+3)(x-1)^2+3c(x-1)+2c+3=0$, (4) implies $27+9(c+3)(y-1)+9c(y-1)^2+(2c-3)(y-1)^3=0$. The norm of y is consequently 1, equal to that of all other elements.

Moreover D has no vanishing minor, since:

If a vanishing n by n determinant D has a vanishing $n-1$ by $n-1$ minor M then either the rows or the columns containing M are dependent.

This follows from Jacobi's theorem on compound determinants, or directly by remarking that unless the columns containing M are dependent, the remaining column of D depends on them and thus its elements partake of the (or a) dependence between the rows of M.

If M is said to belong to the element of D that does not lie on the rows and columns containing M, then the preceding proposition says that an element whose minor vanishes lies on a row or column of such elements. Denoting by K and L the set of rows and the set of columns that are not of this kind, we see:

In a vanishing n by n determinant D there is a unique set K of rows and a unique set L of columns such that an $n-1$ by $n-1$ minor of D vanishes if and only if it is either crossed by all rows of K or by all columns of L.

There exist determinants with arbitrarily prescribed K and L, except that if one set is empty (i.e., if the rank of D is less than $n-1$) so is the other. To see this for given nonempty K and L delete one row of K and one column of L and choose the remaining matrix nonsingular; then prescribe arbitrarily a dependence with nonzero coefficients between the rows of K and a similar dependence for L.

Finally, we note:

For every nonzero characteristic p there exist vanishing 3 by 3 Vandermonde minors composed of qth roots of unity without vanishing subminors. For some characteristics, q may even be a prime, e.g., 17 *for* $p=239$, 13 *for* $p=3$, 11 *for* $p=23$, *or* 7 *for* $p=2$; *in the last case the elements have equal norm in a cubic extension of*

$$F=\{0, 1\}.$$

Consider

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^3 \\ 1 & \epsilon^3 & \epsilon^9 \end{vmatrix} = \epsilon(\epsilon - 1)(\epsilon^2 - 1)(\epsilon^3 - 1)(\epsilon^3 + \epsilon + 1)$$

and

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^3 \\ 1 & \epsilon^4 & \epsilon^{12} \end{vmatrix} = \epsilon(\epsilon - 1)(\epsilon^3 - 1)(\epsilon^4 - 1)(\epsilon^4 + \epsilon + 1).$$

For nonzero characteristic and $\epsilon^3+\epsilon+1=0$ or $\epsilon^4+\epsilon+1=0$, ϵ is a root of unity. For these ϵ no minor of D_1 (of D_2) vanishes if $\epsilon\neq 1$ (respectively, $\epsilon\neq -2$). For $p=239$ ($p=23$) take $\epsilon=-23$ (resp. 4); then $\epsilon^3+\epsilon+1=0$ and $\epsilon^{17}=1$ (resp. $\epsilon^{11}=1$). For $p=3$, $\epsilon^4+\epsilon+1=0$, $\epsilon\neq 1$, implies $\epsilon^{13}=1$. For $p=2$, $\epsilon^3+\epsilon+1=0$ implies $\epsilon^7=1$ and that the norm of ϵ is 1 in $F^*=F(\epsilon)$, $F=\{0, 1\}$.

It is easy to see that 7 is the smallest prime q for which a vanishing 3 by 3 Vandermonde minor composed of qth roots of unity and without vanishing subminors exists, that the only such minor (for $q=11$ as well as $q=7$), up to trivial row and column operations, is D_1, and that the characteristic must be 23 resp. 2.

References

1. T. S. Motzkin, *Bemerkung über Singularitäten gewisser mit Lücken behafteter Potenzreihen*, Math. Ann. vol. 109 (1933) pp. 95–100.

2. ———, *Power series with gaps*, Proc. Amer. Math. Soc. vol. 11 (1960) p. 875.

3. N. Tschebotareff, in A. Ostrowski, *Mathematische Miszellen*. VII: *Über Singularitäten gewisser mit Lücken behafteten Potenzreihen*, Jber. Deutsch. Math. Verein. vol. 35 (1926) pp. 269–280.

University of California, Los Angeles