DETERMINANTS WHOSE ELEMENTS HAVE EQUAL NORM\textsuperscript{1}

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The following theorem on vanishing 3 by 3 determinants I proved previously \cite{1} in the case where $F$ is the field $R$ of reals; the three proofs given there rely on special properties of the complex domain.

Let $F$ be a field of arbitrary characteristic, $F^*$ an extension of degree 2 of $F$. The conjugate of an element $\alpha$ of $F^*$ is denoted by $\bar{\alpha}$; the norm of $\alpha$ is $\alpha \bar{\alpha}$.

**Theorem 1.** A vanishing 3 by 3 determinant of elements of $F^*$ of equal norm has two proportional rows or columns.

Let $\nu \neq 0$, where $\nu$ is the common norm. Then if the terms of the determinant are $A, B, C, A', B', C'$, with

\begin{equation}
A + B + C = A' + B' + C',
\end{equation}

the product of the 9 elements is

\begin{equation}
ABC = A'B'C'.
\end{equation}

From (1) we have $\bar{A} + \bar{B} + \bar{C} = \bar{A'} + \bar{B'} + \bar{C'}$; multiplying by (2) and dividing by $\nu^3$ we obtain

\begin{equation}
AB + AC + BC = A'B' + A'C' + B'C'.
\end{equation}

Hence $A, B, C$ equal $A', B', C'$ in some order. Each of the 6 orders leads immediately to the proportionality of two rows or columns.

The above theorem, in its specialization to minors of Vandermonde determinants composed of $q$th roots of unity in $R^*$, was used in \cite{1} for the proof of a theorem on power series without terms whose subscript belongs to one of 3 residue classes modulo an arbitrary integer $q$. In \cite{2} I showed that the corresponding theorem for 4 residue classes is false for $q = 6$. This suggests that

There exist vanishing 4 by 4 minors of the form $|e^{i\theta}|$, $e^q = 1$, $e$ in $R^*$, without proportional rows or columns.

Indeed, examining the counterexample in \cite{2} in the light of the proof in \cite{1} we obtain the determinant
where $\epsilon$ is a primitive 6th root of unity, and $D$ has no proportional rows or columns. It is known [3] that such determinants do not exist for any degree if $q$ is prime.

However, $D$ has exactly 4 vanishing 3 by 3 minors, viz. those in rows 1, 2, 4; these rows are therefore dependent. The existence of vanishing 4 by 4 determinants (even other than Vandermonde minors) composed of roots of unity in $R^*$ and without dependent rows or columns is still in question. But if the elements are only required to have equal norm we can show:

There exist vanishing 4 by 4 determinants whose elements have norm 1 in $R^*$ and none of whose minors is 0.

Replacing the two last elements of $D$ by $x$ and $y$ we obtain an equation $ax + by + c = 0$ that is also fulfilled by $x = 1, y = -1$. Hence when $x$ traces the unit circle $U$, $y$ traces a circle $V$ meeting $U$ in two points. If the 14 first elements of $D$ are now slightly moved on $U$, so that all 2 by 2 minors involving them only become nonzero, then $a, b, c$ and $V$ move somewhat. Since $V$ continues to meet $U$, values $x$ and $y$ of norm 1 can still be supplied; but now no 3 by 3 minor can vanish, as this would imply the vanishing of a 2 by 2 minor not involving $x$ and $y$. Nor can the product of all 2 by 2 minors involving $x$ or $y$ or both vanish for all small changes as indicated, since it would have to vanish identically; none of these minors, however, vanishes both for $x = -\epsilon, y = -\epsilon^2$ and for $x = 1, y = -1$.

As in [1] Theorem 1 implies

Theorem 2. A matrix of rank 2 of elements of $F^*$ of equal norm consists either of all rows or of all columns of two matrices of rank 1.

Theorem 1 does not hold for general extension fields. We have, e.g.:

There exist vanishing 3 by 3 determinants without proportional rows or columns, composed of elements of equal norm of a cubic extension $F^*$ of the field $F = R_0(\epsilon)$ where $\epsilon^3 + \epsilon = -1$ and $R_0$ is the field of rationals.

The determinant

$$
D = \begin{vmatrix}
1 & 1 & 1 \\
1 & \epsilon & x \\
1 & y & \epsilon^2
\end{vmatrix} = 2 + x + y - xy
$$

where $a_j = 1, 3, 4, 5, b_k = 1, 2, 4, 5,$
vanishes if and only if

\[(4) \quad (x - 1)(y - 1) = 3,\]

but has no proportional rows or columns unless \((x, y)\) or \((y, x) = (e, e^2)\). If \(f(x) = x^3 + cx^2 + (c - 3)x - 1\) is irreducible over \(F\), i.e., if \(f(x) \not= 0\) for \(x = \pm 1, \pm e, \pm e^2\), then \(F^* = F(x)\) with \(f(x) = 0\) is a cubic extension in which the norm of \(x\) is \(-f(0) = 1\). Since \((x - 1)^2 + (c+3)(x-1)^2 + 3c(x-1) + 2c + 3 = 0\), (4) implies \(27 + 9(c+3)(y-1) + 9c(y-1)^2 + (2c-3)(y-1)^3 = 0\). The norm of \(y\) is consequently 1, equal to that of all other elements.

Moreover \(D\) has no vanishing minor, since:

If a vanishing \(n\) by \(n\) determinant \(D\) has a vanishing \(n-1\) by \(n-1\) minor \(M\) then either the rows or the columns containing \(M\) are dependent.

This follows from Jacobi's theorem on compound determinants, or directly by remarking that unless the columns containing \(M\) are dependent, the remaining column of \(D\) depends on them and thus its elements partake of the (or a) dependence between the rows of \(M\).

If \(M\) is said to belong to the element of \(D\) that does not lie on the rows and columns containing \(M\), then the preceding proposition says that an element whose minor vanishes lies on a row or column of such elements. Denoting by \(K\) and \(L\) the set of rows and the set of columns that are not of this kind, we see:

In a vanishing \(n\) by \(n\) determinant \(D\) there is a unique set \(K\) of rows and a unique set \(L\) of columns such that an \(n-1\) by \(n-1\) minor of \(D\) vanishes if and only if it is either crossed by all rows of \(K\) or by all columns of \(L\).

There exist determinants with arbitrarily prescribed \(K\) and \(L\), except that if one set is empty (i.e., if the rank of \(D\) is less than \(n-1\)) so is the other. To see this for given nonempty \(K\) and \(L\) delete one row of \(K\) and one column of \(L\) and choose the remaining matrix nonsingular; then prescribe arbitrarily a dependence with nonzero coefficients between the rows of \(K\) and a similar dependence for \(L\).

Finally, we note:

For every nonzero characteristic \(p\) there exist vanishing 3 by 3 Vandermonde minors composed of \(qth\) roots of unity without vanishing subminors. For some characteristics, \(q\) may even be a prime, e.g., 17 for \(p = 239\), 13 for \(p = 3\), 11 for \(p = 23\), or 7 for \(p = 2\); in the last case the elements have equal norm in a cubic extension of

\[F = \{0, 1\}.\]
Consider

\[ D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^3 \\ 1 & \epsilon^3 & \epsilon^9 \end{vmatrix} = \epsilon (\epsilon - 1)(\epsilon^2 - 1)(\epsilon^3 - 1)(\epsilon^3 + \epsilon + 1) \]

and

\[ D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^3 \\ 1 & \epsilon^4 & \epsilon^{12} \end{vmatrix} = \epsilon (\epsilon - 1)(\epsilon^3 - 1)(\epsilon^4 - 1)(\epsilon^4 + \epsilon + 1). \]

For nonzero characteristic and \( \epsilon^3 + \epsilon + 1 = 0 \) or \( \epsilon^4 + \epsilon + 1 = 0 \), \( \epsilon \) is a root of unity. For these \( \epsilon \) no minor of \( D_1 \) (of \( D_2 \)) vanishes if \( \epsilon \neq 1 \) (respectively, \( \epsilon \neq -2 \)). For \( p = 239 \) (\( p = 23 \)) take \( \epsilon = -23 \) (resp. 4); then \( \epsilon^3 + \epsilon + 1 = 0 \) and \( \epsilon^{17} = 1 \) (resp. \( \epsilon^{11} = 1 \)). For \( p = 3 \), \( \epsilon^4 + \epsilon + 1 = 0 \), \( \epsilon \neq 1 \), implies \( \epsilon^{19} = 1 \). For \( p = 2 \), \( \epsilon^3 + \epsilon + 1 = 0 \) implies \( \epsilon^7 = 1 \) and that the norm of \( \epsilon \) is 1 in \( F^* = F(\epsilon) \), \( F = \{0, 1\} \).

It is easy to see that 7 is the smallest prime \( q \) for which a vanishing 3 by 3 Vandermonde minor composed of \( q \)th roots of unity and without vanishing subminors exists, that the only such minor (for \( q = 11 \) as well as \( q = 7 \)), up to trivial row and column operations, is \( D_1 \), and that the characteristic must be 23 resp. 2.

REFERENCES


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