GENERATING FUNCTIONS FOR FORMAL POWER SERIES IN NONCOMMUTING VARIABLES

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In [1] we stated that knowledge of the coefficients in the formal power series for \( \log e^x e^y \) with \( xy \neq yx \), could be translated into knowledge of the coefficients in \( \log e^{f(x)} e^{g(y)} \) for any power series \( f(x) \) and \( g(y) \) with \( f(0) = g(0) = 0 \). In the following we prove a generalization of this statement, valid for any formal power series in any number of noncommuting variables.

Let \( A(x_1, \ldots, x_n) \) be an arbitrary formal power series in the noncommuting variables \( x_1, \ldots, x_n \). It can be thought of as an element of the free associative ring generated by \( x_1, \ldots, x_n \) over an arbitrary ring which contains its coefficients. For each positive integer \( m \) let \( V_{nm} \) be the set of ordered \( m \)-tuples \((s_1, \ldots, s_m)\) with each \( s_i \) a positive integer no greater than \( n \) and no two consecutive \( s_i \) equal. Let \( a_{s_1 \ldots s_m}(i_1, \ldots, i_m) \), with each \( i_j \) a positive integer, and \((s_1, \ldots, s_m)\) in \( V_{nm} \), be the coefficient of \( x_1^{i_1} \cdots x_n^{i_m} \) in \( A(x_1, \ldots, x_n) \). Thus, if \( a_0 \) is the constant term, we can write \( A(x_1, \ldots, x_n) \) uniquely as

\[
A(x_1, \ldots, x_n) = a_0 + \sum_{m=1}^{\infty} \sum_{(s_1, \ldots, s_m) \in V_{nm}} \sum_{j=1}^{m} \sum_{i_j=1}^{s_j} a_{s_1 \ldots s_m}(i_1, \ldots, i_m) x_1^{i_1} \cdots x_n^{i_m}.
\]

Now, for each positive integer \( m \) and each \((s_1, \ldots, s_m)\) in \( V_{nm} \) we define a correspondence \( T_{s_1 \ldots s_m} \) between \( A(x_1, \ldots, x_n) \) and a generating function in commuting variables \( z_1, \ldots, z_m \) as follows:

\[
T_{s_1 \ldots s_m} : A(x_1, \ldots, x_n) \rightarrow \sum_{j=1}^{m} \sum_{i_j=1}^{s_j} a_{s_1 \ldots s_m}(i_1, \ldots, i_m) z_1^{i_1} \cdots z_m^{i_m}.
\]

This correspondence can be thought of as a linear transformation which takes \( ax_1^{i_1} \cdots x_n^{i_m} \) into \( az_1^{i_1} \cdots z_m^{i_m} \) for all positive \( i_1, \ldots, i_m \), and takes all the other monomials into 0.

We shall prove that this correspondence obeys the following rule.

**Theorem.** Let \( f_1(x), \ldots, f_n(x) \) be any formal power series with constant terms equal to 0. If \( A(x_1, \ldots, x_n) \) is a formal power series in

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noncommuting variables $x_1, \ldots, x_n$ and

$$T_{z_1, \ldots, z_m} : A(x_1, \ldots, x_n) \rightarrow A_{z_1, \ldots, z_m}(z_1, \ldots, z_m),$$

then

$$T_{z_1, \ldots, z_m} : A(f_1(x_1), \ldots, f_n(x_n)) \rightarrow A_{z_1, \ldots, z_m}(f_1(z_1), \ldots, f_m(z_m)).$$

This theorem points out a natural method for handling results about the coefficients in a formal power series in noncommuting variables. There are many ways of choosing a set of generating functions for these coefficients. But the method used in the theorem has the distinct advantage of transforming the set of generating functions into a new set in the same way that the related formal power series are transformed into each other. We have restricted ourselves to transformations with a vanishing constant term for reasons of simplicity, and also because we wished to keep the result strictly formal. Transformations with nonzero constant terms would involve us in considerations of infinite sums of the coefficients.

Following the proof of the theorem we give an example of its application.

The proof of the theorem is a simple exercise in manipulating sums. The only sums in which convergence might be a problem are finite.

We assume that $A(x_1, \ldots, x_n)$ has the expansion (1). Denote the coefficient of $x^k$ in the $i$th power of $f_i(x)$ by $f_{sh}^{(i)}$:

$$\{f_{s}(x)\}^{i} = \sum_{h=0}^{\infty} f_{sh}^{(i)} x^h.$$

Then $A(f_1(x_1), \ldots, f_n(x_n))$ has the expansion

$$a_0 + \sum_{m=1}^{\infty} \sum_{(s_1) \in V_{nm}} \sum_{j=1}^{\infty} \sum_{i_1=1}^{\infty} a_{s_1, \ldots, s_m}(i_1, \ldots, i_m) \cdot \sum_{k=1}^{m} \sum_{h_s=i_k}^{\infty} f_{s_1 h_1}^{(i_1)} \cdots f_{s_m h_m}^{(i_m)} x_{s_1}^{h_1} \cdots x_{s_m}^{h_m}.$$

Therefore the coefficient of $x_{s_1}^{h_1} \cdots x_{s_m}^{h_m}$ in $A(f_1(x_1), \ldots, f_n(x_n))$ is

$$\sum_{j=1}^{m} \sum_{i_j=1}^{h_j} a_{s_1, \ldots, s_m}(i_1, \ldots, i_m) f_{s_1 h_1}^{(i_1)} \cdots f_{s_m h_m}^{(i_m)}.$$

This coefficient is well defined because the sums have a finite number of terms.

It follows that $T_{z_1, \ldots, z_m}$ takes $A(f_1(x_1), \ldots, f_n(x_n))$ into
This last expression is just \( A_{i_1 \cdots i_m} (f_{i_1}(z_1), \ldots, f_{i_m}(z_m)) \), as one can see by comparing it with the right-hand expression in (2). This proves the theorem.

We can now apply this theorem to the formal power series for \( \log F(x)G(y) \) with \( F(0) = G(0) = 1 \). Since we have only two variables, \( V_{nm} \) has only two elements: \((1, 2, 1, 2, \cdots)\) denoting monomials beginning with a power of \( x \), and \((2, 1, 2, 1, \cdots)\) denoting monomials beginning with a power of \( y \). In order to avoid this cumbersome notation, which is necessary only when we have many variables, we will use the subscript \( x \) instead of 1212 \cdots and the subscript \( y \) instead of 2121 \cdots.

Let \( b_x(i_1, \cdots, i_m) \) denote the coefficient of \( x^{i_1}y^{i_2} \cdots \) in \( \log F(x)G(y) \), with \( b_y(i_1, \cdots, i_m) \) similarly defined. We want to find an expression for

\[
B_x(z_1, \cdots, z_m) = \sum_{i_1=1}^{m} \sum_{i_2=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} b_x(i_1, \cdots, i_m) z_1^{i_1} \cdots z_m^{i_m},
\]

and for \( B_y(z_1, \cdots, z_m) \), the generating function of \( b_y(i_1, \cdots, i_m) \).

We shall use the following results from [1]. Let \( a_x(i_1, \cdots, i_m) \) and \( a_y(i_1, \cdots, i_m) \) denote the coefficients in \( \log e^{x}e^{y} \), and let \( A_x(z_1, \cdots, z_m) \) and \( A_y(z_1, \cdots, z_m) \) denote their generating functions. Theorem 2 of [1] states that

\[
A_x(z_1, \cdots, z_m) = \sum_{i=1}^{m} z_i e^{m' z_i} \prod_{j \neq i} (e^{z_i} - 1)(e^{z_i} - e^{z_j})^{-1}
\]

and

\[
A_y(z_1, \cdots, z_m) = \sum_{i=1}^{m} z_i e^{m'' z_i} \prod_{j \neq i} (e^{z_i} - 1)(e^{z_i} - e^{z_j})^{-1}
\]

where \( m' = \lfloor m/2 \rfloor \) and \( m'' = \lfloor (m-1)/2 \rfloor \), with \( \lfloor t \rfloor \) denoting the greatest integer in \( t \).

In order to apply the theorem of this paper we define \( f(x) \) and \( g(y) \) by

\[
F(x) = e^{f(x)}, \quad G(y) = e^{g(y)}
\]

and note that \( f(0) = g(0) = 0 \). Then we can transform the generating
functions for the coefficients in \( \log e^{e^y} \) into the generating functions for the coefficients in \( \log e^{e^y} = \log F(x)G(y) \) directly:

\[
B_x(z_1, \ldots, z_m) = A_x(f(z_1), g(z_2), \ldots, [f, g](z_m))
\]

and

\[
B_y(z_1, \ldots, z_m) = A_y(g(z_1), f(z_2), \ldots, [g, f](z_m))
\]

where \([f, g](z_m)\) denotes \(f(z_m)\) if \(m\) is odd and \(g(z_m)\) if \(m\) is even.

Applying (5) and (6) to (3) we get

\[
B_x(z_1, \ldots, z_m) = \sum \frac{m-m'}{i=1} \left\{ \frac{F(z_{2i-1})}{m-m'} \log F(z_{2i-1}) \prod_{j=1; j \neq i}^{m-m'} \frac{F(z_{2j-1}) - 1}{F(z_{2i-1}) - F(z_{2j-1})} \right. \\
\left. \cdot \prod_{j=1}^{m-m'} \frac{G(z_{2j}) - 1}{F(z_{2i}) - G(z_{2j})} + m-m' \sum_{i=1}^{m-m'} \left\{ \frac{G(z_{2i})}{m-m'} \log G(z_{2i}) \right. \\
\left. \cdot \prod_{j=1; j \neq i}^{m-m'} \frac{F(z_{2j-1}) - 1}{G(z_{2i}) - F(z_{2j-1})} \right. \\
\left. \cdot \prod_{j=1}^{m-m'} \frac{G(z_{2j}) - 1}{F(z_{2i}) - G(z_{2j})} \right. \\n\right.
\]

Applying (5) and (7) to (4) we see that \(B_y(z_1, \ldots, z_m)\) can be obtained from (8) by interchanging \(F\) and \(G\) throughout and then replacing the exponent \(m'\) in the first term of each sum by \(m''\).

Since \(F(0) = 1\) the power series for \(\log F(z)\) is

\[
\log F(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (F(z) - 1)^k.
\]

Thus, an example of (8) is

\[
B_x(z_1, z_2) = F(z_1)(G(z_2) - 1) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{(F(z_1) - 1)^k - (G(z_2) - 1)^k}{F(z_1) - G(z_2)}
\]

\[
- \log G(z_2),
\]

as the reader can verify.

The generating function (8) shows that \(b_x(i_1, \ldots, i_m)\) is left fixed by any permutation of the odd indexed \(i_j\) and any permutation of the even indexed \(i_j\). For example, \(b_x(i_1, i_2, i_3, i_4, i_5) = b_x(i_5, i_4, i_1, i_2, i_3)\). The same is true of \(b_y(i_1, \ldots, i_m)\). If \(F(z) = G(z)\) then any permutation of the \(i_j\) leaves the coefficients unchanged. If \(F(z) = G(z)\) and \(m\) is odd (so that \(m' = m''\)) then \(b_x(i_1, \ldots, i_m) = b_y(i_1, \ldots, i_m)\).

**Reference**


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