ON PROJECTIVE REPRESENTATIONS OF CERTAIN FINITE GROUPS[1]

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Introduction. In this paper it is proved that the irreducible projective representations of the group G of automorphisms of a Lie algebra of classical type constructed in [2] remain irreducible and inequivalent when restricted to the subgroup G_0 of G generated by the one-parameter subgroups $\{\exp(ad\xi e_\alpha)\}$ where α ranges over the set of roots of $\mathfrak{L}$ with respect to a fixed Cartan subalgebra, and ξ is taken from the prime field Ω_0 in Ω. The proofs of irreducibility and inequivalence given in [2] apply without change to G_0 in case Ω_0 is infinite, so there is no problem unless Ω_0 is the prime field of p elements for a prime $p>0$, and in this case an entirely different argument seems to be required.[2]

When Ω_0 is finite, the group G_0 is finite, and can be identified at least in certain cases with one of the finite linear groups introduced by Chevalley [1]. The results of this paper exhibit a family of irreducible projective representations of these groups, while in another paper [3], some results on the degrees of these representations are obtained.[3] The next problem to be investigated in this connection is whether the representations obtained in this paper give all the irreducible projective representations of the groups G_0.

1. **Preliminary results.** Familiarity with the paper [2] is assumed.

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[2] The referee has pointed out that all the irreducible rational projective representations of the group G have been determined by algebraic-geometric methods in Séminaire C. Chevalley, Paris 1956–1958, Exposés 15 and 16. Not all the irreducible representations obtained by Chevalley, however, yield irreducible projective representations upon restriction to the subgroup G_0. For example, let $G=PSL(2, \Omega)$, where Ω is an algebraically closed field of characteristic $p>0$. Then $G_0=PSL(2, \Omega_0)$. It follows easily from Chevalley's classification of the rational irreducible representations of $SL(2, \Omega)$ (Exposé 20, pp. 20-11 ff.) that G has irreducible rational projective representations of arbitrarily high degree. On the other hand, G_0, being a finite group, has at most a finite number of inequivalent irreducible projective representations in the field Ω. This result can be proved using the methods of I. Schur, J. Reine Angew. Math. vol. 127 (1904) pp. 20–50, or K. Asano and K. Shoda, Compositio Math. vol. 2 (1935) pp. 230–240, especially §1. The full connection between the representations constructed in [2] and those determined in the Séminaire C. Chevalley remains to be determined.

[3] See note added in proof at the end of this paper.

We begin by recalling, with some minor changes, some of the principal notations in [2].

Ω algebraically closed field of characteristic $p>7$;

Ω_0 prime field in Ω;

$\mathfrak{L}$ Lie algebra of classical type over Ω;

$\mathfrak{H}$ a fixed Cartan subalgebra of $\mathfrak{L}$;

$\alpha, \beta, \cdots$ roots of $\mathfrak{L}$ with respect to $\mathfrak{H}$;

$e(\alpha)$ a fixed basis element for the root space $\mathfrak{L}_\alpha$, $\alpha \neq 0$;

$\Delta = \{\alpha_1, \cdots, \alpha_l\}$ a maximal simple system of roots of $\mathfrak{L}$ with respect to $\mathfrak{H}$;

M a fixed irreducible restricted right $\mathfrak{L}$-module;

λ maximal weight of M;

x_+, x_- fixed maximal and minimal vectors, respectively, in M.

Our main task is to find a manageable set of generators for M. We begin with the remark that for all $v \in M$, x, y in $\mathfrak{L}$, we have

$$(vx)y - (vy)x = v[xy], \tag{1}$$

because M is a right $\mathfrak{L}$-module. From [2] we know that M is spanned over Ω by x_+ together with vectors

$$x_+ e(\gamma_1) \cdots e(\gamma_r), \qquad \gamma_i < 0,$$

and that for each negative root γ,

$$e(\gamma) = \xi[\cdots[e(-\alpha_{i_1})e(-\alpha_{i_2})] \cdots e(-\alpha_{i_s})], \qquad \alpha_{i_j} \in \Delta, \ \xi \in \Omega.$$

Combining these facts we conclude that M is spanned over Ω by x_+ together with the *vector monomials*

$$v = x_+ e(-\alpha_{i_1}) \cdots e(-\alpha_{i_s}), \qquad \alpha_{i_j} \in \Delta, \ s \geqq 0. \tag{2}$$

We define the ***rank*** of the expression (2) for v to be the ordered $l$tple of non-negative integers $(\rho_i) = (\rho_1, \cdots, \rho_l)$, where ρ_i counts the number of indices j, $1 \leqq j \leqq s$, for which $i_j = i$ in (2); in other words ρ_i counts the multiplicity of $e(-\alpha_i)$ as a "factor" in (2). We shall call ρ the ***rank*** of the vector monomial v, and denote it by $\rho(v)$. We shall prove shortly that the rank of v is well defined; until then, when we speak of a vector monomial v of rank ρ, we mean that v can be expressed as a vector monomial (2) of rank ρ. The maximal vector x_+ is counted as a vector monomial of rank 0. The utility of the notion of rank comes from the fact that the ranks can be linearly ordered lexicographically. We define $\rho = (\rho_i) < \rho' = (\rho_i')$ if $\rho \neq \rho'$ and if the first nonvanishing difference $\rho_i' - \rho_i$ is positive. An arbitrary vector w is called a ***rank vector*** of rank ρ if $w \neq 0$ and if w is a linear combination of vector monomials of rank ρ. Two vectors of the same rank also have the same weight in the sense of [2], but the converse is not necessarily true.

We shall denote by ϵ_i, $1 \leqq i \leqq l$, the ith "unit vector" with a 1 in the ith position and zeros elsewhere. Our first lemma can now be stated as follows.

(1.1) LEMMA. *Let v be a rank vector of rank ρ, and let $\alpha_i \in \Delta$. If $ve(\alpha_i) \neq 0$ then $ve(\alpha_i)$ is a rank vector and $\rho(ve(\alpha_i)) = \rho - \epsilon_i$; while if $ve(-\alpha_i) \neq 0$, $\rho(ve(-\alpha_i)) = \rho + \epsilon_i$.*

PROOF. It is sufficient to prove the result in case v is a vector monomial (2) of rank ρ. We require the fact that because Δ is a simple system, the difference of two roots α and β in Δ is either zero or is not a root. Therefore

$$[e(-\alpha_k)e(\alpha_i)] = \begin{cases} 0 & \text{if } k \neq i, \\ h_{\alpha_k} \in \mathfrak{H} & \text{if } k = i. \end{cases} \tag{3}$$

Now let v be given by (2) and let $ve(\alpha_i) \neq 0$. Then the number of factors s in (2) is not zero, and we have

$$\begin{aligned} ve(\alpha_i) &= x_+ e(\alpha_i) e(-\alpha_{i_1}) \cdots e(-\alpha_{i_s}) + x_+[e(-\alpha_{i_1}) \cdots e(-\alpha_{i_s}), e(\alpha_i)] \\ &= \sum_{k=1}^{s} x_+ e(-\alpha_{i_1}) \cdots e(-\alpha_{i_{k-1}})[e(-\alpha_{i_k})e(\alpha_i)]e(-\alpha_{i_{k+1}}) \cdots e(-\alpha_{i_s}), \end{aligned}$$

since $x_+ e(\alpha_i) = 0$. By (3) each term in the last sum is either zero or a multiple of some vector monomial of rank $\rho - \epsilon_i$. This proves the first assertion. The second is immediate from the definition of rank.

(1.2) COROLLARY. *Let v_1 and v_2 be rank vectors such that $v_i e(\alpha) \neq 0$, $i = 1, 2$, for some α such that either $\pm\alpha \in \Delta$. Then $\rho(v_1) < \rho(v_2)$ if and only if $\rho(v_1 e(\alpha)) < \rho(v_2 e(\alpha))$.*

(1.3) LEMMA. *Rank vectors of different ranks are linearly independent.*

PROOF. Suppose there exist rank vectors $w_1, \cdots, w_t$, $t > 1$, such that $\rho(w_1) > \cdots > \rho(w_t)$ and $w_1 + \cdots + w_t = 0$. Because M is irreducible, there exists a sequence of integers $j_1, \cdots, j_n$, $1 \leqq j_k \leqq l$, such that

$$w_1^* = w_1 e(\alpha_{j_1}) \cdots e(\alpha_{j_n})$$

is a maximal vector, and by the Corollary to Theorem 1 of [2], w_1^* is a nonzero multiple of x_+. By Lemma 1.1 it follows that

$$\rho(w_1) - \epsilon_{j_1} - \cdots - \epsilon_{j_n} = 0.$$

Because $\rho(w_i) < \rho(w_1)$ if $i > 1$, we have

$$w_i e(\alpha_{j_1}) \cdots e(\alpha_{j_n}) = 0,$$

and obtain the impossible conclusion

$$0 = (w_1 + \cdots + w_t)e(\alpha_{j_1}) \cdots e(\alpha_{j_n}) = w_1 e(\alpha_{j_1}) \cdots e(\alpha_{j_n}) = w_1^* \neq 0.$$

Therefore our original assumption that a relation $w_1 + \cdots + w_t = 0$, $t > 1$, could exist was incorrect, and Lemma 1.3 is proved.

(1.4) COROLLARY. *The rank of a vector monomial is well defined; in other words it is impossible for a vector v to have two expressions* (2) *of different ranks.*

(1.5) COROLLARY. *There is a unique maximal rank ρ^*. Any vector of rank ρ^* is a multiple of the minimal vector x_-.*

PROOF. The first statement is immediate by Lemma 1.3 and the fact that M is finite dimensional. For the second, let v be a rank vector which is not a multiple of x_-. Then by the Corollary to Theorem 1 of [2], this time applied to $(0:\mathfrak{U}_-)$, we conclude that there exist integers $k_1, \cdots, k_s$, $1 \leqq k_i \leqq l$, such that

$$ve(-\alpha_{k_1}) \cdots e(-\alpha_{k_s}) = \xi x_-, \qquad \xi \neq 0.$$

This result combined with Lemma 1.1 implies that $\rho(v) \leqq \rho(x_-)$, and Corollary 1.5 is proved.

The next Lemma is a refinement of Lemma (II. 2.1) of [2]. We recall that the group G is generated by the automorphisms.

$$\sigma = \sigma(\alpha, \xi) = \exp(ad\xi e(\alpha)), \qquad \xi \in \Omega,$$

where α is a root of $\mathfrak{L}$ with respect to $\mathfrak{H}$. The projective representation F maps σ onto the transformation $F(\sigma)$ given by

$$x_+ e(-\alpha_{i_1}) \cdots e(-\alpha_{i_r}) \to x_+ e(-\alpha_{i_1})^\sigma \cdots e(-\alpha_{i_r})^\sigma$$

if $\alpha > 0$, and

$$x_- e(\alpha_{j_1}) \cdots e(\alpha_{j_s}) \to x_- e(\alpha_{j_1})^\sigma \cdots e(\alpha_{j_s})^\sigma$$

if $\alpha < 0$.

(1.6) LEMMA. *Let v be a rank vector in M, and let $\sigma = \sigma(\alpha_i, \xi)$, where either $\pm\alpha_i \in \Delta$. Then*

$$vF(\sigma) = v + \xi ve(\alpha_i) + v^*, \qquad \xi \in \Omega,$$

where v^* is a sum of rank vectors all of rank $< \rho(v) - \epsilon_i$ if $\alpha_i \in \Delta$, and a sum of rank vectors all of rank $> \rho(v) + \epsilon_i$ if $-\alpha_i \in \Delta$.

PROOF. First suppose that $\alpha_i \in \Delta$. We may assume that v is a vector monomial $v = x_+ e(-\alpha_{j_1}) \cdots e(-\alpha_{j_r})$. By the argument of Lemma (II.2.1) of [2], we have

$$vF(\sigma) = v + \xi v e(\alpha_i) + \sum_{k \geq 2} \xi^k v_k,$$

where v_k, $k \geqq 2$, is a linear combination of vectors of the form

$$x_+ \{ e(-\alpha_{j_1})(ad\,e(\alpha_i))^{h_1} \} \cdots \{ e(-\alpha_{j_r})(ad\,e(\alpha_i))^{h_r} \}, \tag{4}$$

where $\sum h_i = k$. It is immediate by (3) that such a vector is either zero or a rank vector of rank $\rho(v) - k\epsilon_i < \rho(ve(\alpha_i))$, and upon setting $v^* = \sum_{k \geq 2} \xi^k v_k$, the first assertion of the Lemma is established. The second assertion is proved similarly, starting from a vector v of the form $x_- e(\alpha_{j_1}) \cdots e(\alpha_{j_r})$, $\alpha_{j_i} \in \Delta$, and observing that such a vector is a rank vector by Lemma 1.1 and Corollary 1.5. We shall omit the rest of the proof of the second statement.

(1.7) LEMMA. *Let w be a nonzero vector in M such that $wF(\sigma) = w$ for every $\sigma = \sigma(\alpha_i, 1)$, $\alpha_i \in \Delta$. Then w is a maximal vector. Similarly, if $wF(\sigma) = w$ for every $\sigma = \sigma(-\alpha_i, 1)$, $\alpha_i \in \Delta$, then w is a minimal vector.*

PROOF. Again we shall prove only the first assertion. Let $w = w_1 + \cdots + w_t$, where the w_i are rank vectors such that $\rho(w_1) > \cdots > \rho(w_t)$ if $t > 1$. Let $\sigma = \sigma(\alpha_i, 1)$; then by Lemma 1.6 and the hypothesis of Lemma 1.7 we have

$$wF(\sigma) = w = w + \sum w_k e(\alpha_1) + \sum w_k^*,$$

where each w_k^* is a sum of rank vectors of rank less than the rank of $w_k e(\alpha_i)$. Then

$$\sum w_k e(\alpha_i) + \sum w_k^* = 0, \tag{5}$$

and if $w_1 e(\alpha_i) \neq 0$ then this term is the only rank vector in (5) of rank $\rho(w_1) - \epsilon_i$, and we contradict Lemma 1.3. Therefore $w_1 e(\alpha_i) = 0$ for all α_i in Δ, and w_1 is a maximal vector. Because w_1 has the greatest rank among all the w_i, we have $w_2 = \cdots = w_t = 0$. Hence $w = w_1$ is a maximal vector, and Lemma 1.7 is proved.

2. **The main theorem.** The main theorem of the paper can be stated as follows.

THEOREM. *Let F be the projective representation of G associated with the irreducible restricted $\mathfrak{L}$-module M. Let G_0 be the subgroup of G generated by the automorphisms $\sigma(\alpha, \xi)$ where ξ is taken from the prime field Ω_0 in Ω, and α ranges over the set of roots of $\mathfrak{L}$ with respect to a fixed*

Cartan subalgebra $\mathfrak{H}$. *Then the restriction* F_0 *of* F *to the subgroup* G_0 *is an irreducible projective representation of* G_0. *Moreover if* F *and* F' *are two projective representations of* G *associated with the irreducible restricted* $\mathfrak{L}$*-modules* M *and* M', *and if* S *is a vector space isomorphism of* M *onto* M' *such that* $SF'(\sigma)=F(\sigma)S$ *for all generators* $\sigma=\sigma(\alpha, \xi)$, $\xi\in\Omega_0$, *of* G_0, *then* M *and* M' *are* $\mathfrak{L}$*-isomorphic.*

Proof. We prove first that F_0 is irreducible. Let $N\neq 0$ be an Ω-subspace of M which is invariant with respect to all $F(\sigma)$, $\sigma\in G_0$. Let v be a nonzero element of N, and write $v=\sum_1^t v_i$, where the v_i are rank vectors such that $\rho(v_1)> \cdots >\rho(v_t)$. There exist integers $i_1, \cdots, i_s$, $1\leqq i_j\leqq l$, such that $v_1e(\alpha_{i_1})\cdots e(\alpha_{i_s})$ is a nonzero multiple of x_+. By successive applications of Lemma 1.6, we obtain

$$ve(\alpha_{i_1})\cdots e(\alpha_{i_s}) + v^* = \sum_1^t v_ie(\alpha_{i_1})\cdots e(\alpha_{i_s}) + v^* \in N,$$

where $v_1e(\alpha_{i_1})\cdots e(\alpha_{i_s})$ is the unique term of highest rank in the expression. It follows that

$$\sum_2^t v_ie(\alpha_{i_1})\cdots e(\alpha_{i_s}) + v^* = 0$$

and that $x_+=\xi v_1e(\alpha_{i_1})\cdots e(\alpha_{i_s})\in N$. Similarly $x_-\in N$.

By the same reasoning we see that for every rank vector in M of the form

$$v = x_-e(\alpha_{i_1})\cdots e(\alpha_{i_s}), \tag{6}$$

N contains a vector $v+v^*$ where v^* is a sum of rank vectors all of rank less than $\rho(v)$. The space M has a basis $\{v_i\}$ consisting of vectors of the form (6), and corresponding to this basis we have a set of vectors $\{w_i=v_i+v_i^*\}$ in N. We prove that the vectors w_i are linearly independent. If we have a relation of linear dependence

$$\sum \xi_iw_i = 0,$$

with some $\xi_i\neq 0$, then there will exist vectors $w_{i_j}=v_{i_j}+v_{i_j}^*$, $1\leqq j\leqq r$, with nonzero coefficients ξ_{i_j}, and with the rank of the v_{i_j} as large as possible. Applying Lemma 1.3, we obtain $\sum_1^r \xi_{i_j}v_{i_j}=0$, contrary to the assumption that the $\{v_i\}$ are linearly independent. Therefore N contains a basis of M, and we have proved that M is irreducible relative to G_0.

In order to prove the second assertion, it is sufficient by Theorem 1 of [2], to prove that M and M' have the same maximal weight. From the hypothesis that

$$SF'(\sigma) = F(\sigma)S, \qquad \sigma \in G_0, \tag{7}$$

we obtain

$$x_+SF'(\sigma) = x_+F(\sigma)S = x_+S, \qquad \sigma = \sigma(\alpha_i, 1), \qquad \alpha_i \in \Delta.$$

By Lemma 1.7 applied to M', we see that x_+S is a maximal vector in M', and by a similar argument, x_-S is a minimal vector in M'. The concept of rank is meaningful in both M and M', and we shall prove, after some preliminary steps, that S preserves rank.

(2.1) LEMMA. $\rho(x_-S) = \rho(x_-)$.

PROOF. Suppose first that $\rho(x_-) < \rho(x_-S)$. We shall then prove that if v is an arbitrary rank vector in M, and if $vS = \sum_{j\geq 0} w_j'$, where the w_j' are rank vectors in M' such that $\rho(w_0') < \rho(w_1') < \cdots$, then $\rho(v) < \rho(w_0')$. We have the result for the vector x_- of maximal rank, and we may assume as an induction hypothesis that the result is true for all v_1 such that $\rho(v_1) > \rho(v)$. We may also assume that w_0' is not a minimal vector in M', otherwise v is a minimal vector in M, and the result is known. There exists a root $\alpha_i \in \Delta$ such that $w_0' e(-\alpha_i) \neq 0$. Applying (7) and Lemma 1.6 to v and $\sigma = \sigma(-\alpha_i, 1)$, we obtain

$$(v + ve(-\alpha_i) + v^*)S = vS + \sum w_j' e(-\alpha_i) + \sum (w_j')^*, \tag{8}$$

where v^* is a sum of rank vectors all of rank $> \rho(v) + \epsilon_i$, while $w_0' e(-\alpha_i)$ is the unique term of minimal rank on the right-hand side after vS has been cancelled. Applying the induction hypothesis to $ve(-\alpha_i)$ and v^*, we conclude that $\rho(v) + \epsilon_i < \rho(w_0' e(-\alpha_i))$, and it follows that $\rho(v) < \rho(w_0')$ as required. The result just established, however, implies that $\rho(x_+) < \rho(x_+S)$ which is impossible. Therefore the hypothesis that $\rho(x_-) < \rho(x_-S)$ is untenable, and $\rho(x_-) \geqq \rho(x_-S)$. Similarly $\rho(x_-) \leqq \rho(x_-S)$, and Lemma 2.1 is proved.

(2.2) LEMMA. *For any rank vector v in M, $vS = \sum_{j\geq 0} w_j'$, where the w_j' are rank vectors in M' such that $\rho(v) \leqq \rho(w_0') < \rho(w_1') < \cdots$.*

PROOF. By Lemma 2.1, the result is true for v of maximal rank, and we may assume it for all v_1 such that $\rho(v_1) > \rho(v)$. As in the proof of Lemma 2.1, find $\alpha_i \in \Delta$ such that $w_0' e(-\alpha_i) \neq 0$, and write down the equation (8). Again we may apply the induction hypothesis to the left side, to conclude that $\rho(v) + \epsilon_i \leqq \rho(w_0' e(-\alpha_i))$ and hence $\rho(v) \leqq \rho(w_0')$.

Similarly we can prove that $vS = \sum_{j\geq 0} w_j''$, where the w_j'' are rank vectors in M' such that $\rho(v) \geqq \rho(w_0'') > \cdots$. Combining our results, and applying Lemma 1.3, we deduce that for all rank vectors v in M, vS is a rank vector in M' and $\rho(vS) = \rho(v)$.

Now we can prove that M and M' have the same maximal weight.

First suppose $x_+e(-\alpha_i) \neq 0$. Then from $x_+F(\sigma(-\alpha_i, 1))S = x_+SF'(\sigma(-\alpha_i, 1))$ and Lemma 1.6 we obtain

$$(x_+ + x_+e(-\alpha_i) + x_+^*)S = x_+S + x_+Se(-\alpha_i) + (x_+S)^*.$$

Because S preserves rank we can apply Lemma 1.3 to get

$$x_+e(-\alpha_i)S = (x_+S)e(-\alpha_i). \tag{9}$$

Now apply $F'(\sigma(\alpha_i, 1))$ to both sides of (9). This yields

$$x_+e(-\alpha_i)F(\sigma(\alpha_i, 1))S = (x_+S)e(-\alpha_i)F'(\sigma(\alpha_i, 1)),$$

and we obtain

$$[x_+e(-\alpha_i) + x_+e(-\alpha_i)e(\alpha_i) + (x_+e(-\alpha_i))^*]S$$
$$= x_+Se(-\alpha_i) + x_+Se(-\alpha_i)e(\alpha_i) + (x_+Se(-\alpha_i))^*.$$

Equating terms of equal rank we have

$$\lambda(h_{\alpha_i})x_+S = \lambda'(h_{\alpha_i})x_+S,$$

where λ and λ' are the maximal weights of M and M' respectively. Finally suppose $x_+e(-\alpha_i) = 0$; then $\lambda(h_{\alpha_i}) = 0$, and we obtain $x_+Se(-\alpha_i) = 0$, so that $\lambda'(h_{\alpha_i}) = 0$. We have proved that $\lambda(h_{\alpha_i}) = \lambda'(h_{\alpha_i})$ for all $\alpha_i \in \Delta$. Because the h_{α_i} span $\mathfrak{H}$, we conclude that $\lambda = \lambda'$, and the theorem is proved.

Added in proof. We take this opportunity to correct an error in [3]. In that paper the assertion on p. 141 "$m_\alpha = m_\alpha'$" following the proof of Theorem 2 is false, and invalidates the subsequent construction of the example, although formula (9) is correct as it stands. An example to show that Weyl's formula does not always hold at characteristic p can be constructed as follows. Let $\mathfrak{L}$ be the simple Lie algebra of type A_2 over Ω of characteristic $p \geqq 5$, viewed as the algebra of linear transformations of trace zero on a 3-dimensional vector space M_0. Let α_1 and α_2 be a maximal simple system of roots with respect to a Cartan subalgebra, and let v_0 be a maximal vector in M_0 such that $v_0e_{\alpha_2} \neq 0$. Let $v_0^p = v_0 \otimes \cdots \otimes v_0$ (p times) in the tensor algebra $T(M_0)$ on M_0. Then $v = v_0^p e_{-\alpha_2}$ is a maximal vector in $T(M_0)$ of weight λ such that $\lambda(h_{\alpha_1}) = 1$, $\lambda(h_{\alpha_2}) = p-2$. The irreducible restricted $\mathfrak{L}$-module M whose maximal weight is λ is a composition factor of $v\mathfrak{U}$, and since $v\mathfrak{U}$ is contained in the space of symmetric p-tensors, we have $\dim M \leqq (p+1)(p+2)/2$. On the other hand, formula (9) of [3] asserts that for the associated module V of M we have

$$\dim V = p^2 - 1 > \dim M$$

if $p \geqq 5$.

References

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