# ON APPROXIMATE DERIVATIVES

CASPER GOFFMAN AND C. J. NEUGEBAUER

**Introduction.** The notion of approximate derivative was introduced by A. Denjoy in 1916 [**1**; see also **4**], and has important applications to integration theory and to the relation between partial and total derivatives. It is natural to ask whether the main properties of derivatives are also possessed by approximate derivatives. Indeed, Khintchine [**3**] has shown that Rolle's theorem holds for approximate derivatives, and Tolstoff [**5**] has shown that every approximate derivative is of Baire class 1. It readily follows that every approximate derivative has the Darboux property. Moreover, Tolstoff [**6**] has shown that every approximately differentiable function is differentiable except possibly on a nowhere dense set.

Since the proofs given by Khintchine and Tolstoff are long and difficult, and because of the intrinsic value of the subject, it seems proper that a simple, unified treatment should be available. It is hoped that the present work accomplishes this purpose.

**Baire class one.** Throughout the paper $I_0$ will denote the interval $[0, 1]$.

THEOREM 1. *Assume that* $f\colon I_0 \to R$ *has an approximate derivative* $f'_{ap}$ *everywhere on* $I_0$. *Then* $f'_{ap}$ *is of Baire class* 1.

PROOF. We will show that $f'_{ap}$ is the limit of a convergent interval function, which by [**2**] completes the proof.

For $I$ a subinterval of $I_0$, let

$$A(I; k) = \left\{(x, y): x, y \in I \text{ and } \frac{f(x) - f(y)}{x - y} > k\right\}.$$

Define an interval function $F(I)$ by

$$F(I) = \sup\left\{k: \frac{|A(I; k)|}{|I|^2} > \frac{1}{2}\right\}.$$

We show that $F$ converges to $f'_{ap}$, i.e., for $x_0 \in I_0$ and $\{I_n\}$ a sequence of intervals in $I_0$ such that $x_0 \in \bigcap_{n\geq 1} I_n$, $\lim |I_n| = 0$, we have

$$\lim F(I_n) = f'_{ap}(x_0). \tag{1}$$

Presented to the Society, November 26, 1960; received by the editors January 30, 1960.

In order to prove (1), let $f'_{ap}(x_0)=\zeta$, and let $1/8>\epsilon>0$ be given. There is a positive integer $N$ such that

$$\left|\frac{f(x)-f(x_0)}{x-x_0}-\zeta\right|<\epsilon \tag{2}$$

on a set $E$ for which $|E\cap I_n|/|I_n|>1-\epsilon$, $n\geqq N$.

Let $n\geqq N$ be fixed. For $x\in E_n$, where $E_n=E\cap I_n$, let

$$E_x=\{y: y\in E_n \text{ and } |x-x_0|<8|x-y|\}.$$

We assert that

$$x\in E_n,\ y\in E_x \quad\text{implies}\quad \left|\frac{f(x)-f(y)}{x-y}-\zeta\right|<17\epsilon. \tag{3}$$

To prove (3), we may assume that $x\neq x_0$, say $x>x_0$. We will further assume that $y>x$, the case $y<x$ being treated analogously. We may also suppose $x_0=0$, $f(x_0)=0$.

Since $x<8(y-x)$, $x=\tau y$ where $0<\tau<8/9$. From (2) we get

$$-\epsilon<-\frac{f(x)}{x}+\zeta<\epsilon,\qquad -\epsilon<-\zeta+\frac{f(y)}{y}<\epsilon$$

and adding,

$$-2\epsilon<\frac{f(y)}{y}-\frac{f(x)}{x}<2\epsilon.$$

Thus, $\tau f(y)-2\tau\epsilon y<f(x)<\tau f(y)+2\tau\epsilon y$. From this, we infer

$$(1-\tau)f(y)-2\tau\epsilon y<f(y)-f(x)<(1-\tau)f(y)+2\tau\epsilon y.$$

Since $y-x=(1-\tau)y>0$,

$$\frac{f(y)}{y}-\frac{2\tau\epsilon}{1-\tau}<\frac{f(y)-f(x)}{y-x}<\frac{f(y)}{y}+\frac{2\tau\epsilon}{1-\tau}\cdot$$

Since $\zeta-\epsilon<f(y)/y<\zeta+\epsilon$ and $2\tau/(1-\tau)<16$, we obtain

$$\zeta-\epsilon-16\epsilon<\frac{f(y)-f(x)}{y-x}<\zeta+\epsilon+16\epsilon,$$

from which (3) follows.

By an easy calculation, we obtain

$$x\in E_n \quad\text{implies}\quad |E_x|>|I_n|\cdot(1-\epsilon)\cdot 3/4. \tag{4}$$

From (3) and (4) it follows that the set of points $(x, y)\in I_n\times I_n$ for which

$$\left|\frac{f(x) - f(y)}{x - y} - \zeta\right| < 17\epsilon$$

has measure $|I_n|^2(1-\epsilon)^2 \cdot 3/4 > |I_n|^2/2$. Hence, $|F(I_n) - \zeta| < 17\epsilon$, $n \geqq N$, and the proof is complete.

**Darboux property.** The purpose of this section is to show that an approximate derivative has the Darboux property, and that the mean value theorem for "approximate differentiation" is valid.

DEFINITION. A subset $E$ of $I_0$ will be termed *d-closed* iff $E$ contains every point of positive upper density of $E$.

LEMMA 1. *If $f: I_0 \to R$ is approximately continuous on $I_0$, then the set*

$$E = \{x : x \in I_0 \text{ and } f(x) \geqq f(0)\}$$

*is d-closed.*

PROOF. This is an immediate consequence of the approximate continuity of $f$.

LEMMA 2. *Let $f: I_0 \to R$ have an approximate derivative $f'_{ap}$ everywhere on $I_0$. If $f'_{ap} \geqq 0$ on $I_0$, then $f$ is monotone nondecreasing*[1] *on $I_0$.*

PROOF. We first assume $f'_{ap} > 0$ on $I_0$. Let $E = \{x : x \in I_0$ and $f(x) \geqq f(0)\}$, $0 < \alpha < 1$, and let $C$ be a set satisfying the following conditions:

(i) $C \subset E$;

(ii) $x', x'' \in C$ and $x' < x''$ implies $|E \cap [x', x'']| / (x'' - x') \geqq \alpha$. Since $f'_{ap} > 0$ on $I_0$, such sets $C$ exist. Let $\mathcal{R}$ be the collection of all such sets $C$. If $\mathcal{R}$ is partially ordered by set inclusion, it is readily seen that every linearly ordered subset of $\mathcal{R}$ has an upper bound in $\mathcal{R}$. By Zorn's lemma, $\mathcal{R}$ has a maximal element $K$.

Let $\beta = \sup K$. We assert that $\beta \in K$. For this we show $x \in K$, $x < \beta$, implies $|E \cap [x, \beta]| / (\beta - x) \geqq \alpha$ and $\beta \in E$. If $x \in K$, $x < \beta$, there is a sequence $\{x_n\} \subset K$ such that $x < x_n \leqq x_{n+1}$ and $\lim x_n = \beta$. Then

$$\alpha \leqq \lim \frac{|E \cap [x, x_n]|}{x_n - x} = \frac{|E \cap [x, \beta]|}{\beta - x}.$$

Thus $\beta$ is a point of positive upper density of $E$, and hence by Lemma 1, $\beta \in E$.

Next we show that $\beta = 1$. For, if $\beta < 1$, there would exist $\gamma > \beta$ such that $\gamma \in E$ and $|E \cap [\beta, \gamma]| / (\gamma - \beta) \geqq \alpha$. Hence $\gamma \in K$, which is impossible. Thus $\beta = 1$, and $f(1) \geqq f(0)$.

If $f'_{ap} \geqq 0$ on $I_0$, then for any $\epsilon > 0$, the function $h_\epsilon(x) = f(x) + \epsilon x$ has a positive approximate derivative on $I_0$. Hence $f(0) \leqq f(1) + \epsilon$, and since $\epsilon > 0$ was arbitrary, $f(0) \leqq f(1)$.

---

[1] In [6] a more general theorem is given whose proof is long and involved.

Applying the above argument to any $[x', x'']\subset I_0$, we obtain $f(x')\leqq f(x'')$.

COROLLARY 1. *Under the hypothesis of Lemma 2, $f'_{ap}$ is the ordinary derivative of $f$.*

PROOF. This is clear.

COROLLARY 2. *If $f\colon I_0\to R$ has an approximate derivative $f'_{ap}$ everywhere on $I_0$, and if there exists a derivative $\phi'$ such that $\phi'\geqq f'_{ap}(\phi'\leqq f'_{ap})$ on $I_0$, then $f'_{ap}$ is the ordinary derivative of $f$ on $I_0$ [3].*

PROOF. If $\phi'\geqq f'_{ap}$, then $\phi'-f'_{ap}\geqq 0$ on $I_0$. Since $\phi'-f'_{ap}$ is the approximate derivative of $\phi-f$ on $I_0$, by Corollary 1 the proof is complete.

Let $f\colon I_0\to R$ have an approximate derivative $f'_{ap}$ everywhere on $I_0$.

DEFINITION. $f'_{ap}$ has *property D* on $I_0$ iff $f'_{ap}$ has the Darboux property on $I_0$, i.e., if $0\leqq\alpha<\beta\leqq 1$ and $\eta$ is between $f'_{vp}(\alpha)$, $f'_{ap}(\beta)$, there exists $\xi$, $\alpha<\xi<\beta$, such that $f'_{ap}(\xi)=\eta$.

DEFINITION. $f'_{ap}$ has *property M* on $I_0$ iff for $0\leqq\alpha<\beta\leqq 1$ there is $\alpha<\xi<\beta$ such that $f(\beta)-f(\alpha)=f'_{ap}(\xi)(\beta-\alpha)$.

THEOREM 2. *The properties D and M are equivalent for approximate derivatives.*

PROOF. Let $f\colon I_0\to R$ have an approximate derivative $f'_{ap}$ on $I_0$.

(1) $M$ implies $D$. We only need show that, if $0\leqq\alpha<\beta\leqq 1$ and $f'_{ap}(\alpha)<0<f'_{ap}(\beta)$, then there exists $\xi$, $\alpha\leqq\xi\leqq\beta$, such that $f'_{ap}(\xi)=0$. We may assume that $f(\beta)\geqq f(\alpha)$, the case $f(\beta)<f(\alpha)$ being treated similarly. If $f(\beta)=f(\alpha)$, property $M$ assures the existence of such a $\xi$. If $f(\beta)>f(\alpha)$, we have a point $\eta$, $\alpha<\eta<\beta$, such that $f(\eta)=f(\alpha)$, since $f$ (being approximately continuous) satisfies property $D$. Again application of property $M$ assures the existence of a $\xi$.

(2) $D$ implies $M$. Consider $0\leqq\alpha<\beta\leqq 1$. It suffices to suppose $f(\alpha)=f(\beta)$, and to show there is a $\xi\in(\alpha,\beta)$ such that $f'_{ap}(\xi)=0$. If $f'_{ap}>0$ on $(\alpha,\beta)$, then, by Lemma 2, $f(\alpha)<f(\beta)$. Similarly, $f'_{ap}<0$ on $(\alpha,\beta)$ is impossible. Hence there exist $\eta',\eta''\in(\alpha,\beta)$ such that $f'_{ap}(\eta')\geqq 0$, $f'_{ap}(\eta'')\leqq 0$. Property $D$ yields the existence of a $\xi\in(\alpha,\beta)$ such that $f'_{ap}(\xi)=0$.

THEOREM 3. *Assume that $f\colon I_0\to R$ has an approximate derivative $f'_{ap}$ everywhere on $I_0$. Then $f'_{ap}$ possesses the property D.*

PROOF. Suppose that $f'_{ap}$ does not satisfy the property $D$. We may then assume that $f'_{ap}(0)<0$, $f'_{ap}(1)>0$, and there is no $\xi\in(0,1)$ such that $f'_{ap}(\xi)=0$.

Let $E^+=\{x\colon f'_{ap}(x)>0\}$, $E^-=\{x\colon f'_{ap}(x)<0\}$. Then $I=E^+\cup E^-$.

If $Q$ is a component of either $E^+$ or $E^-$, then $Q$ is either a single point or else a closed interval.

To prove this, assume that $Q$ is not a single point, and $Q$ is a component of $E^+$. Then $Q$ is an interval. Let $a$, $b$ be the endpoints of $Q$. Since by Lemma 2, $f$ is monotone increasing on $(a, b)$, we have, in view of the approximate continuity of $f$, that $f$ is monotone increasing on $[a, b]$. Hence $f'_{ap}(a) \geqq 0$, $f'_{ap}(b) \geqq 0$. Since $I_0 = E^+ \cup E^-$, we infer that $f'_{ap}(a) > 0$, $f'_{ap}(b) > 0$. Thus $Q = [a, b]$.

By considering, if necessary, a subinterval of $I_0$, we may suppose that 0 is a component of $E^-$, 1 is a component of $E^+$. Let $\{Q^+\}$, $\{Q^-\}$ be the components of $E^+$, $E^-$, respectively, which are intervals, and let $\{Q\} = \{Q^+\} \cup \{Q^-\}$. Then two distinct elements in $\{Q\}$ are disjoint. Hence $P = I_0 - \bigcup_{Q \in \{Q\}} Q^0$ is perfect and $f'_{ap}$ has no point of continuity in $P$ relative to $P$. However, this contradicts Theorem 1.

**Approximate and ordinary derivatives.** The following theorem is also contained in [**5**].

THEOREM 4. *Let $f: I_0 \to R$ have an approximate derivative $f'_{ap}$ everywhere on $I_0$. Let $E = \{x: f'(x)$ exists$\}$. Then, for every subinterval $I$ of $I_0$, $I \cap E$ contains an interval.*

PROOF. Suppose there exists a subinterval $I$ of $I_0$ such that $I \cap E$ contains no interval. We may assume that $I = I_0$. Let $E^+ = \{x: f'_{ap}(x) \geqq 0\}$, $E^- = \{x: f'_{ap}(x) < 0\}$. By Corollary 1, every component of $E^+$, $E^-$ is a single point, and since $I_0 = E^+ \cup E^-$, both $E^+$ and $E^-$ are dense in $I_0$. Let $\alpha > 0$ and let $E_\alpha^- = \{x: f'_{ap}(x) \leqq -\alpha\}$. We assert that $E_\alpha^-$ is dense in $I_0$. As proof consider $g(x) = f(x) + \alpha \cdot x$. Then $g$ satisfies the hypothesis of $f$, and $A = \{x: g'(x)$ exists$\}$ contains no interval. Hence $E_g^- = \{x: g'_{ap}(x) \leqq 0\}$ is dense in $I_0$. Since $g'_{ap}(x) = f'_{ap}(x) + \alpha$, we deduce $E_g^- = E_\alpha^-$.

Since $E^+$ and $E_\alpha^-$ are dense in $I_0$, $f'_{ap}$ has no point of continuity in $I_0$, contradicting Theorem 1.

## BIBLIOGRAPHY

1. A. Denjoy, *Mêmoire sur la totalisation des nombres dérivês nonsommables*, Ann. Sci. Ecole Norm. Sup. vol. 33 (1916) pp. 127–222.

2. A. Gleyzal, *Interval functions*, Duke Math. J. vol. 8 (1941) pp. 223–230.

3. A. Khintchine, *Recherches sur la structure des fonctions mesurables*, Fund. Math. vol. 9 (1927) pp. 217–279 (esp. pp. 243–253).

4. S. Saks, *Theory of the integral*, Warszawa-Lwow, 1937.

5. G. Tolstoff, *Sur la dérivêe approximative exacte*, Rec. Math. (Mat. Sbornik) N.S. vol. 4 (1938) pp. 499–504.

6. ———, *Sur quelques propriétés des fonctions approximativement continues*, Rec. Math. (Mat. Sbornik) N.S. vol. 5 (1939) pp. 637–645.

PURDUE UNIVERSITY