ALMOST UNIFORM CONVERGENCE VERSUS POINTWISE CONVERGENCE

JOHN W. BRACE[1]

In many an example of a function space whose topology is the topology of almost uniform convergence it is observed that the same topology is obtained in a natural way by considering pointwise convergence of extensions of the functions on a larger domain [**1**; **2**]. This paper displays necessary conditions and sufficient conditions for the above situation to occur.

Consider a linear space $G(S, F)$ of functions with domain S and range in a real or complex locally convex linear topological space F. Assume that there are sufficient functions in $G(S, F)$ to distinguish between points of S. Let S_β denote the closure of the image of S in the cartesian product space $\times\{\overline{g(S)}: g\in G(S, F)\}$. Theorems 4.1 and 4.2 of reference [**2**] give the following theorem.

THEOREM. *If $g(S)$ is relatively compact for every g in $G(S, F)$, then pointwise convergence of the extended functions on S_β is equivalent to almost uniform converqence on S.*

When almost uniform convergence is known to be equivalent to pointwise convergence on a larger domain the situation can usually be converted to one of equivalence of the two modes of convergence on the same domain by means of Theorem 4.1 of [**2**]. In the new formulation the following theorem is applicable.

In preparation for the theorem, let $B(S, R)$ denote all bounded real valued functions on S which are uniformly continuous for the uniformity which $G(S, F)$ generates on S. $G(S, F)$ will be called a *full linear space* if for every f in $B(S, R)$ and every g in $G(S, F)$ the function fg obtained from their pointwise product is a member of $G(S, F)$. (S, G) denotes S with the weakest topology such that each g in $G(S, F)$ is continuous, while $(S, G)\cup\{\infty\}$ denotes the one point compactification of (S, G).

THEOREM. *If $G(S, F)$ is a full linear space in which pointwise convergence and almost uniform convergence are equivalent then (S, G) is*

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either (i) *compact, or* (ii) *locally compact and each g in* $G(S, F)$ *can be continuously extended over* $(S, G)\cup\{\infty\}$ *such that* $g(\infty)=0$.

PROOF. Let CS be the compactification of (S, G) obtained by taking the closure of the image of S in $\times\{\overline{f(S)}:f\in B(S, R)\}$. Assume (S, G) is not compact and that there is an s_0 in $CS-S$, a net $\{s_\alpha\}$ in S, and a function g in $G(S, F)$ such that the net $\{s_\alpha\}$ converges to s_0 but the net $\{g(s_\alpha)\}$ does not converge to 0. There exists a net $\{f_\delta\}$ in $B(S, R)$ converging pointwise on S to the function which is identically one while all continuous extensions of the members of the net have the value zero at s_0. This gives a contradiction in that the net $\{f_\delta g\}$ in $G(S, F)$ converges pointwise to g but not almost uniformly. Thus when (S, G) is not compact each g in $G(S, F)$ can be continuously extended over CS so that $g(s)=0$ for every s in $CS-S$. Since CS is the Hausdorff completion of (S, G), there is only one point in $CS-S$. Thus (S, G) is locally compact and the proof completed.

COROLLARY. *A uniform space E is compact if and only if pointwise convergence and almost uniform convergence are equivalent in* $C(E, R)$, *all uniformly continuous real valued functions on E.*

PROOF. If E is compact, see Theorem 4.2 of [**2**]. For the converse, observe that each f in $C(E, R)$ is bounded, Theorem 1.3 of [**2**]. Thus fg is in $C(E, R)$ whenever f and g are, and $C(E, R)$ is a full linear space. Since $C(E, R)$ contains nonzero constant functions, conclusion (ii) of the second theorem is not possible and E must be compact.

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UNIVERSITY OF MARYLAND