NOTE ON THE HOMOTOPY PROPERTIES OF THE COMPONENTS OF THE MAPPING SPACE X^{S^p}

S. S. KOH[1]

1. **Introduction.** Let X be a topological space and S^p be the polarized p-sphere with a fixed pole y_0. Following G. W. Whitehead [**10**], we shall denote by $G^p(X)$ the mapping space X^{S^p}, which is the totality of (continuous) maps of S^p into X endowed with compact-open topology. Let $\pi: G^p(X) \rightarrow X$ be defined by $\pi(f) = f(y_0)$, $(f \in G^p(X))$, and let $F^p(X, x) = \pi^{-1}(x)$ for each $x \in X$. Consider now the mapping space $B(X)$ consisting of all the maps of y_0 into X. There is a natural map $p: G^p(X) \rightarrow B(X)$ defined by $p(f) = f | y_0$ for every $f \in G^p(X)$. It is well known (cf. [**3**, pp. 83–84]) that p has the path lifting property. Clearly, the space X can be identified with $B(X)$ in a natural way. The map π is then identified with p. Consequently $\pi: G^p(X) \rightarrow X$ is a fibre map of $G^p(X)$ onto X having the absolute covering homotopy property [**3**, p. 82]. For each $x \in X$, the fibre in $G^p(X)$ over x is $F^p(X, x)$. The arc components of $F^p(X, x)$ are elements of the pth homotopy group $\pi_p(X, x)$ of X at x. Denote by $G^p_\alpha(X)$ the arc component of $G^p(X)$ which contains $\alpha = F^p_\alpha(X, x) \in \pi_p(X)$ (cf. [**10**]). If X is arcwise connected, then $G^p_\alpha(X)$ is also a fibre space over X. The restriction $\pi_\alpha = \pi | G^p_\alpha(X)$ is a fibre map of $G^p_\alpha(X)$ onto X. The homotopy properties of the various components $G^p_\alpha(X)$ of $G^p(X)$ have been studied by M. Abe (Jap. J. Math. vol. 16 (1940) pp. 169–176), G. W. Whitehead [**10**] and S. T. Hu [**2**]. The present note may be regarded as a continuation of these studies.

2. ***H*-space and H_*-space.** In what follows, we shall denote $G^p(X)$ by G^p and $F^p(X, x)$ by F^p whenever no confusion is likely to arise.

Let X be a topological space which admits a continuous multiplication $\mu(x, x') = x \cdot x'$. If $f: S \rightarrow X$ is a map of a space S into X, we denote by $x \cdot f$ the transformation defined by $(x \cdot f)(s) = x \cdot f(s)$ for each $s \in S$. Clearly $x \cdot f$ is a map (i.e. it is continuous).

By an H-space we mean a topological space X with a given continuous multiplication which has a homotopy unit $e \in X$ (see e.g. [**3**, pp. 80–81]).

Received by the editors February 17, 1960.

[1] The writer held an assistantship under the Air Force Contract AF 49 (638)-179.

(2.1) THEOREM. *If X is an arcwise connected H-space, then $G^p_\alpha(X)$ and $G^p_\beta(X)$ have the same homotopy type for arbitrary α and β in $\pi_p(X)$, $p \geqq 1$.*

PROOF. It suffices to prove that $G^p_\alpha(X)$ and $G^p_0(X)$ have the same homotopy type, for any $\alpha \in \pi_p(X)$. According to [10], it remains to prove that $G^p_\alpha(X)$ admits a (global) cross-section. Choose an element $f \in G^p_\alpha \cap F^p(X, e)$. Then $\pi_\alpha(f) = e$. Define $\phi: X \rightarrow G^p$ by $\phi(x) = x \cdot f$. Then $\phi(e) = e \cdot f \simeq f \in G^p_\alpha$. Since X is arcwise connected, we have $\phi: X \rightarrow G^p_\alpha$. Now, $\pi_\alpha(\phi(x)) = \pi_\alpha(x \cdot f) = x \cdot e$, therefore $\pi_\alpha \phi \simeq \mathrm{id}_X$. Since π_α has the absolute covering homotopy property, there exists a covering homotopy, in particular, there is a map $\psi: X \rightarrow G^p_\alpha$ such that $\pi_\alpha \psi = \mathrm{id}_X$. This proves (2.1).

Following H. Wada [9], we call a topological space X an H_*-space if the following conditions are satisfied:

(i) A continuous multiplication $\mu(x, x') = x \cdot x'$ is defined for each pair of elements x, x' in X.

(ii) There is a fixed element e in X, satisfying

$$x \cdot e = x, \qquad \text{for all } x \in X.$$

(iii) To each $x \in X$, there is an inverse $x^{-1} \in X$, defined continuously by x, such that

$$x \cdot x^{-1} = e, \qquad \text{for all } x \in X.$$

(iv) For each pair of elements x, x' in X, we have

$$x^{-1} \cdot (x \cdot x') = x'.$$

With these conditions Wada was able to prove that

(ii′) e is unique,

(iii′) x^{-1} is uniquely defined by x and $x^{-1} \cdot x = e$,

(v) $(x^{-1})^{-1} = x$ and, consequently, $x \cdot (x^{-1} \cdot x') = x'$, for arbitrary x and x' in X.

We remark that an H_*-space need not to be an H-space.

The following theorem resembles a construction of Wada [9], where he deals with mapping space of an H_*-space into itself.

(2.2) THEOREM. *Let X be an H_*-space. The mapping space $G^p(X)$ is homeomorphic to $X \times F^p(X, e)$ for each $p \geqq 1$.*

PROOF. Let $g \in G^p(X)$ be an arbitrary map of S^p into X. Then $x \cdot g$ is defined and continuous. Hence $x \cdot g \in G^p(X)$. Clearly $g = e \cdot g = x^{-1} \cdot (x \cdot g) = x \cdot (x^{-1} \cdot g)$ for any $x \in X$. Let

$$\phi: G^p(X) \to X \times F^p(X, e),$$

and

$$\psi: X \times F^p(X, e) \to G^p(X),$$

be defined as follows: Let y_0 be the pole of S^p. For each $g \in G^p(X)$, let $\hat{g} = g(y_0) \in X$. Then define

$$\phi(g) = (\hat{g}, \hat{g}^{-1} \cdot g), \qquad (g \in G^p(X))$$

and

$$\psi(x, f) = x \cdot f, \qquad (x \in X, f \in F^p(X, e)).$$

(A) ϕ and ψ are bijective:

For any $g \in G^p(X)$, we have

$$\psi\phi(g) = \psi(\hat{g}, \hat{g}^{-1} \cdot g) = \hat{g} \cdot (\hat{g}^{-1} \cdot g) = g.$$

On the other hand,

$$\begin{aligned} \phi\psi(x, f) = \phi(x \cdot f) &= ((x \cdot f)\hat{}, ((x \cdot f)\hat{})^{-1} \cdot (x \cdot f)) \\ &= (x \cdot \hat{f}, (x \cdot \hat{f})^{-1} \cdot (x \cdot f)) \\ &= (x \cdot e, (x \cdot e)^{-1} \cdot (x \cdot f)) \\ &= (x, x^{-1} \cdot (x, f)) \\ &= (x, f). \end{aligned}$$

Hence both ϕ and ψ are one-to-one, onto.

(B) ϕ and ψ are continuous:

Suppose K be a compact set in S^p and U an open set in X. We shall denote by (K, U) be the subset of $G^p(X)$ consisting of all mappings which send K into U. Let H be an arbitrary neighborhood of $(\hat{g}, \hat{g}^{-1} \cdot g)$. Then $H \supset U_0 \times [(K_1, U_1) \cap \cdots \cap (K_n, U_n)]$ for some open sets $U_0, U_1, \cdots, U_n$ in X and compact sets $K_1, \cdots, K_n$ in S^p. Denote $g(K_i)$ by K_i', then K_i' is compact, $i = 1, 2, \cdots, n$. Corresponding to each $k_i^\alpha \in K_i'$, there exist open sets W_i^α containing $\hat{g}^{-1}$ and V_i^α containing k_i^α such that $W_i^\alpha \cdot V_i^\alpha \subset U_i$, since the multiplication in X is continuous. The collection $\{V_i^\alpha\}$ forms an open covering of K_i'. There is a finite subcovering $\{V_i^{\alpha_1}, \cdots, V_i^{\alpha_{m_i}}\}$ of K_i'. Let $W_i = \bigcap_{j=1}^{m_i} W_i^{\alpha_j}$ and $V_i = \bigcup_{i=1}^{m_i} V^{\alpha_i}$. Then W_i is an open neighborhood of $\hat{g}^{-1}$; V_i is an open neighborhood of K_i' and $W_i \cdot V_i \subset U_i$.

Let $N = (y_0, U_0 \cap W_1^{-1} \cap \cdots \cap W_n^{-1}) \cap (K_1, V_1) \cap \cdots \cap (K_n, V_n)$, where W_i^{-1} denotes, of course, the set $\{w^{-1} \mid w \in W_i\}$. By the continuity of the inverse, N is a neighborhood of g in G^p. It is now readily seen that $\phi(N) \subset H$. This proves the continuity of ϕ.

Next, let $U = (K_1, U_1) \cap (K_2, U_2) \cap \cdots \cap (K_n, U_n)$ be a basic open neighborhood of $\psi(x, f) = x \cdot f$. Then $x \cdot f(K_i) \subset U_i$. By a similar argument as above, one proves that there exist open neighborhoods W_i of x and V_i of $f(K_i)$ such that $W_i \cdot V_i \subset U_i$. Then

$$\psi[(W_1 \cap \cdots \cap W_n) \times ((K_1, V_1) \cap \cdots \cap (K_n, V_n) \cap F^p)] \subset U.$$

Hence ψ is continuous and the proof of (2.2) is completed.

(2.3) COROLLARY. *If X is an arcwise connected H_*-space, then G^p_α and $X \times F^p_\alpha$ are homeomorphic.*

PROOF. Since X is arcwise connected, G^p_α is a fibre space over X. By replacing G^p and G^p_α and π by π_α in the proof of (2.2), we obtain that G^p_α is homeomorphic to $X \times \pi_\alpha^{-1}(e)$. Being a component, G^p_α is connected hence $\pi_\alpha^{-1}(e)$ contains only one component F^p_α. This proves (2.3).

As a by-product of the proof of (2.3) we have:

(2.4) COROLLARY. *Every arcwise connected H_*-space is n-simple, for $n \geqq 1$.*

(2.5) COROLLARY. *If X is an arcwise connected H_*-space, then G^p_α and G^p_β have the same homotopy type for arbitrary α and β in $\pi_p(X)$. Furthermore*

$$\pi_q(G^p_\alpha) \approx \pi_{p+q}(X) + \pi_q(X), \qquad (q \geqq 1).$$

PROOF. Since G. W. Whitehead [10] proved that F^p_α and F^p_β have the same homotopy type for any α and β in $\pi_p(X)$, the first part of (2.5) follows from (2.3). The Hurewicz isomorphism $\pi_q(F^p_\alpha) \approx \pi_{p+q}(X)$ (cf. [10]) completes the proof.

(2.6) COROLLARY. *Let $X = S^r$. Then G^p_α is homeomorphic to $S^r \times F^p_\alpha$ when $r = 1$, 3 or 7. Conversely, if $G^p_{i_r}$ and $S^r \times F^p_0$ have the same homotopy type then $r = 1$, 3 or 7, where $i_r \in \pi_r(S^r)$ is represented by the identity map $S^r \to S^r$.*

PROOF. This follows from Wada [8] and a recent result of Adams [1].

(2.6) PROPOSITION. *If X is a H-space, then for each $\alpha \in \pi_p(X)$,*

$$\pi_q(G^p_\alpha)/\pi_{p+q}(X) \approx \pi_q(X),$$

where $\pi_{p+q}(X)$ is, of course, imbedded in $\pi_q(G^p_\alpha)$ isomorphically.

PROOF. According to G. W. Whitehead [10] (see also [11]), we have the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_q(F^p_\alpha) \arrow[r, "i_*"] \arrow[d, leftrightarrow] & \pi_q(G^p_\alpha) \arrow[r, "j_*"] & \pi_q(G^p_\alpha, F^p_\alpha) \arrow[r, "\partial"] \arrow[d, leftrightarrow, "\pi_*"] \arrow[dr] & \pi_{q-1}(F^p_\alpha) \arrow[r] \arrow[d, leftrightarrow, "H"] & \cdots \\
& \pi_{p+q}(X) & & \pi_q(X) \arrow[r, "\rho_\alpha"'] & \pi_{p+q-1}(X), &
\end{tikzcd}$$

where π_* denotes the isomorphism induced by the projection π, H denotes the Hurewicz isomorphism and ρ_α is defined by $\rho_\alpha(\beta) = -[\alpha, \beta]$. Since ρ_α is always trivial when X is an H-space, (2.7) follows from the exactness of the sequence.

3. **The sphere S^r.** Let $X = S^r$, an r-sphere, then we have the following exact sequence

$$(3.1) \qquad \cdots \xrightarrow{\rho_\alpha} \pi_{p+q}(S^r) \xrightarrow{\mu} \pi_q(G^p_\alpha) \xrightarrow{\nu} \pi_q(S^r) \xrightarrow{\rho_\alpha} \pi_{p+q-1}(S^r) \xrightarrow{\mu} \cdots.$$

The following propositions are fairly obvious.

(3.2) Proposition. *Let $X = S^r$ and $\alpha \in \pi_p(S^r)$. Since $\pi_q(S^r) = 0$ for $q < r$ we have*

$$\pi_q(G^p_\alpha) \approx \pi_{p+q}(S^r), \qquad (q < r-1).$$

(3.3) Corollary. $\pi_1(G^r_\alpha) \approx Z_2$ *for* $r \geqq 3$.

Since $\pi_{r+2}(S^r) \approx Z_2$, for $r \geqq 3$, we have

(3.4) Corollary. $\pi_1(G^{r+1}) \approx Z_2$, $r \geqq 3$.

Denote the image of $\rho_\alpha\colon \pi_q(S^r) \to \pi_{p+q-1}(S^r)$ by J^{p+q-1}_α and the kernel of ρ_α by K^p_α. Denote the image of $\mu\colon \pi_{p+q}(S^r) \to \pi_q(G^p_\alpha)$ by P_α. Then

(3.5) Proposition (Hu) [2]. *For $X = S^r$ and $\alpha \in \pi_p(S^r)$*

(a) $\pi_q(G^p)/P^q_\alpha \approx K^q_\alpha$, $\qquad (q > 1)$,
(b) $\pi_{p+q}(G^p_\alpha)/J^{p+q}_\alpha \approx P^q$, $\qquad (q > 1)$,
(c) $\pi_{r-1}(G^q_\alpha) \approx \pi_{p+r-1}(S^r)/J^{p+r-1}_\alpha$,
(d) $\pi_{r+3}(G^p_\alpha)$ *has a subgroup* $P^{r+3}_\alpha \approx \pi_{p+r+3}(S^r)$, $\qquad (r \geqq 6)$,
(e) $\pi_{r+4}(G^p) \approx \pi_{p+r+4}(S^r)/J^{p+r+4}$, $\qquad (r \geqq 6)$.

Since for $r \geqq 7$, $\pi_{r+4}(S^r) = \pi_{r+5}(S^r) = 0$. It follows that

(3.6) Proposition. If $r > 7$, *for each* $\alpha \in \pi_p(S^r)$,

$$\pi_{r-1}(G^7) \approx \pi_{r-2}(G^8) \approx \cdots \approx 0.$$

And,

(3.7) PROPOSITION. *For $r \geqq 7$, $\alpha \in \pi_p(S^r)$,*

$$\pi_{r+6-p}(G^p) \approx \pi_{r+6-p}(S^r).$$

We now proceed to prove the main theorem of this section. Consider the following sequence

$$(3.8) \qquad \pi_r(S^r) \xrightarrow{\rho_\alpha} \pi_{2r-1}(S^r) \xrightarrow{E} \pi_{2r}(S^{r+1}),$$

where E denotes the Freudenthal suspension. By the delicate suspension theorem, the kernel of E is a cyclic subgroup generated by $[\iota_r, \iota_r]$. If r is even, it is infinite cyclic; if r is odd $\neq 1, 3, 7$, it is cyclic of order 2.

(3.9) LEMMA (HU). *For $X = S^2$ and $\alpha \in \pi_2(S^2)$, we have*

$$\pi_1(G_\alpha^2) \approx Z_{2m},$$

where m is the absolute value of the degree of α.

PROOF. Since $\pi_{2r}(S^{r+1}) = \pi_4(S^3) \approx Z_2$. From (3.8) $\pi_3(S^2)/\mathrm{Ker}\, E \approx Z_2$. Let γ be a generator of the free cyclic group $\pi_3(S^2)$. Then $[\iota_2, \iota_2] = \pm 2$. We can choose γ so that $[\iota_2, \iota_2] = -2\gamma$. Let $\alpha \in \pi_2(S^2)$. By linearity of the Whitehead product $\rho_\alpha(\iota_2) = -[\alpha, \iota_2] = -m[\iota_2, \iota_2] = 2m\gamma$. In other words J_α^3 is generated by $2m\gamma$. From (3.5(c)), we have $\pi_1(G_\alpha^2) \approx Z_{2m}$. This proves (3.9).

(3.10) LEMMA. *For $X = S^4$ and $\alpha \in \pi_4(S^4)$, we have*

$$\pi_3(G_\alpha^4) \approx Z_{24m} + Z_{12},$$

where m is the absolute value of the degree of α.

PROOF. $\pi_{2r}(S^{r+1}) \approx \pi_8(S^5) \approx Z_{24}$ and $\pi_{2r-1}(s^r) = \pi_7(S^4) \approx Z + Z_{12}$. One generator of Ker E is determined as follows:

From a theorem of characteristic map [5, p. 121], that

$$[\iota_4, \iota_4] = 2[q] - \epsilon E[\xi],$$

where $\epsilon = \pm 1$ depends on the convention of orientation, $[q]$ denotes the homotopy class of the Hopf map $q: S^7 \to S^4$ and $[\xi]$ a generator of $\pi_6(S^3)$ represented by the characteristic map $\xi: S^6 \to S^3$ of the fibre bundle Sp(2) over S^7 with Sp(1) as fibre. Hence in $\pi_8(S^5)$ we have

$$E^2[\xi] = \epsilon 2E[q],$$

(E^2 denotes the iterated suspension). This implies that $\pi_8(S^5)$ has $E[q]$ as a generator. Hence

$$\pi_7(S^4)/\mathrm{Ker}\, E \approx Z_{24} + Z_{12}.$$

A similar argument as used in (3.9) yields

$$\pi_3(\overset{4}{G_\alpha}) \approx Z_{24m} + Z_{12}.$$

(3.11) LEMMA. *For $X = S^6$ and $\alpha \in \pi_6(S^6)$, we have*

$$\pi_5(\overset{6}{G_\alpha}) \approx Z_m,$$

where m is the absolute value of the degree of α.

PROOF. Since $\pi_{2r}(S^{r+1}) = \pi_{12}(S^7) = 0$ and $\pi_{2r-1}(S^5) = \pi_{11}(S^6) \approx Z$. Ker $E = J^{11}_\alpha$. Hence we can choose the generator γ of $\pi_{11}(S^6)$ such that $\gamma = -[\iota_6, \iota_6]$, consequently $\rho_\alpha(\iota_6) = m\gamma$, or $\pi_5(G^6_\alpha) \approx Z_m$ by (3.4(c)).

(3.12) LEMMA. *For $X = S^8$ and $\alpha \in \pi_8(S^8)$, we have*

$$\pi_7(G^8) \approx Z_{240m} + Z_{120},$$

where m is the absolute value of the degree of α.

PROOF. $\pi_{2r}(S^{r+1}) = \pi_{16}(S^9) \approx Z_{240}$ and $\pi_{2r-1}(S^r) = \pi_{15}(S^8) \approx Z + Z_{120}$. Since $[\iota_6, \iota_6] = 2[q'] - \epsilon E[\xi']$, where $[q']$ denote the homotopy class represented by the Hopf map $q': S^{15} \to S^8$ and $\xi' \in \pi_{14}(S^7)$ has nonzero Hopf invariant, we have

$$E^2[\xi'] = 2\epsilon E[q'].$$

Using the same argument as in (3.10), one proves (3.12).

(3.13) LEMMA. *For $X = S^{10}$ and $\alpha \in \pi_{10}(S^{10})$, we have*

$$\pi_9(G^{10}) \approx Z_m + Z_2 + Z_2 + Z_2,$$

where m denotes the absolute value of the degree of α.

(3.14) LEMMA. *For $X = S^{12}$ and $\alpha \in \pi_{12}(S^{12})$, we have*

$$\pi_{11}(G^{12}) \approx Z_m + Z_8 + Z_{27} + Z_7,$$

where m denotes the absolute value of the degree of α.

The proof of (3.13) follows from the table in Toda [6] the first row and a similar argument as before; for a proof of (3.14), one uses the third row of the above mentioned table.

(3.15) LEMMA. *For $X = S^{14}$ and $\alpha \in \pi_{14}(S^{14})$, we have*

$$Z_{13}(G^{14}) \approx Z_m + Z_3,$$

where m denotes the absolute value of the degree of α.

PROOF. Since $\pi_{17}(S^{14}) \approx Z + Z_3$ and $\pi_{18}(S^{15}) \approx Z_3$ and the suspension E sends Z into 0 in $\pi_{18}(S^{15})$ (Toda [6]). The proof is immediate.

(3.16) LEMMA. *For $X=S^r$, $\alpha\in\pi_r(S^r)$ and r odd, $\neq 1$, 3, 7. Then*
(a) $\pi_{r-1}(G^r)\approx\pi_{2r-1}(S^r)$ *when α is of even degree,*
(b) $\pi_{r-1}(G^r)\approx\pi_{2r-1}(S^r)/Z_2$ *when α is of odd degree.*

PROOF. It suffices to prove that there is a nonzero element in J_α^{2r-1} when α is of odd degree and $r\neq 1$, 3, 7. In fact, in this case $\rho_\alpha(\iota_r)\neq 0$. (3.16) follows.

(3.17) LEMMA (HU). *Let X be any space. If α, $\beta\in\pi_p(X)$, $\alpha+\beta=0$. Then G_α^p and G_β^p are homeomorphic.*

PROOF. Let $\theta\colon S^p\to S^p$ be a homeomorphism which reverses the orientation and leaves the pole y_0 fixed. Then a homeomorphism h of G_α^p onto G_β^p is given by $h(f)=f\cdot\theta$ for each $f\in G_\alpha^p$.

(3.18) THEOREM. *Let $X=S^r$. Let α, $\beta\in\pi_r(S^r)$. Then for $r=2$, 4, 6, 8, 10, 12, 14, the components G_α^r and G_β^r have the same homotopy type if and only if $\alpha=\pm\beta$. When r is odd $\neq 1$, 3, 7, the components G_α^r and G_β^r are of different homotopy type if* deg $\alpha-$deg β *is odd.*

PROOF. The first part of the theorem follows from Lemmas (3.9) through (3.17). The remaining part follows from the fact that if r is odd then $\pi_p(S^r)$ is finite for $p>n$ [**4**].

(3.19) COROLLARY. *Let $X=S^r$ and α, $\beta\in\pi_r(S^r)$ are of odd and even degree respectively. Then*:

$$\begin{aligned}
&\pi_4(G_\alpha^5) = 0, && \pi_4(G_\beta^5) \approx Z_2,\\
&\pi_8(G_\alpha^9) \approx Z_2+Z_2, && \pi_8(G_\beta^9) \approx Z_2+Z_2+Z_2,\\
&\pi_{10}(G_\alpha^{11}) \approx Z_2+Z_9, && \pi_{10}(G_\beta^{11}) \approx Z_2+Z_2+Z_9,\\
&\pi_{12}(G_\alpha^{13}) = 0, && \pi_{12}(G_\beta^{13}) \approx Z_2,\\
&\pi_{14}(G_\alpha^{15}) \approx Z_2+Z_2 \text{ or } Z_4, && \pi_{14}(G_\beta^{15}) \approx Z_4+Z_2.
\end{aligned}$$

(3.20) PROPOSITION (HU). *When r is even and $\alpha\in\pi_r(S^r)$, $r\neq 0$. Then*

$$\pi_r(G_\alpha^r) \approx \pi_{2r}(S^r)/J_\alpha^{2r}.$$

PROOF. Since $K_\alpha^r=0$, the result follows from (3.4(a)) and (3.4(b)).

(3.21) PROPOSITION. *If $E\colon\pi_p(S^r)\to\pi_{p+1}(S^{r+1})$ is an injection, then for $q+s=p$ and $q>1$*

$$\pi_q(G_\alpha^s)/\pi_p(S^r) \approx \pi_q(S^r),$$

where $\pi_p(S^r)$ is imbedded in $\pi_q(G_\alpha^s)$.

PROOF. Since $E[\alpha, \beta]=0$, $J_\alpha^p \subset \operatorname{Ker} E=0$. From (3.4)(a) and (b), $\pi_q(G_\alpha^s)/\pi_{q+s}(S^r) \approx K_\alpha^q$. But $K_\alpha^q \approx \pi_q(S^r)$. This proves (3.21).

For $q<r$, $\pi_q(S^r)=0$, we have $\pi_q(G_\alpha^s) \approx \pi_p(S^r)$. This reduces to (3.2).

(3.22) COROLLARY. *If* $q+s=p<2r-1$, *then*

$$\pi_q(G_\alpha^s)/\pi_p(S^r) \approx \pi_q(S^r).$$

PROOF. This follows from (3.21).

BIBLIOGRAPHY

1. J. F. Adams, Bull. Amer. Math. Soc. vol. 64 (1958) pp. 277–282.
2. S. T. Hu, Indag. Math. vol. VII (1946) pp. 621–629.
3. ———, *Homotopy theory*, New York, Academic Press, 1959.
4. J.-P. Serre, Ann. of Math. vol. 54 (1951) pp. 425–505.
5. N. E. Steenrod, *The topology of fibre bundles*, Princeton University Press, 1951.
6. H. Toda, C. R. Acad. Sci. Paris vol. 240 (1955) pp. 147–149.
7. ———, C. R. Acad. Sci. Paris vol. 241 (1955) pp. 847–850.
8. H. Wada, Ann. of Math. vol. 64 (1956) pp. 420–435.
9. ———, Tôhoku Math. J. vol. 10 (1958) pp. 143–145.
10. G. W. Whitehead, Ann. of Math. vol. 47 (1946) pp. 460–475.
11. J. H. C. Whitehead, Ann. of Math. vol. 58 (1953) pp. 418–428.

WAYNE STATE UNIVERSITY