NOTE ON THE HOMOTOPY PROPERTIES OF THE COMPONENTS OF THE MAPPING SPACE $X^{Sp}$

S. S. KOH

1. Introduction. Let $X$ be a topological space and $S^p$ be the polarized $p$-sphere with a fixed pole $y_0$. Following G. W. Whitehead [10], we shall denote by $G^p(X)$ the mapping space $X^{S^p}$, which is the totality of (continuous) maps of $S^p$ into $X$ endowed with compact-open topology. Let $\pi: G^p(X) \to X$ be defined by $\pi(f) = f(y_0)$, $(f \in G^p(X))$, and let $F^p(x, x) = \pi^{-1}(x)$ for each $x \in X$. Consider now the mapping space $B(X)$ consisting of all the maps of $y_0$ into $X$. There is a natural map $\hat{p}: G^p(X) \to B(X)$ defined by $\hat{p}(f) = f|_{y_0}$ for every $f \in G^p(X)$. It is well known (cf. [3, pp. 83–84]) that $\hat{p}$ has the path lifting property. Clearly, the space $X$ can be identified with $B(X)$ in a natural way. The map $\pi$ is then identified with $\hat{p}$. Consequently $\pi: G^p(X) \to X$ is a fibre map of $G^p(X)$ onto $X$ having the absolute covering homotopy property [3, p. 82]. For each $x \in X$, the fibre in $G^p(X)$ over $x$ is $F^p(x, x)$. The arc components of $F^p(x, x)$ are elements of the $p$th homotopy group $\pi_p(X, x)$ of $X$ at $x$. Denote by $G^p_\alpha(X)$ the arc component of $G^p(X)$ which contains $\alpha = F^p_\alpha(x, x) \in \pi_p(X)$ (cf. [10]). If $X$ is arcwise connected, then $G^p_\alpha(X)$ is also a fibre space over $X$. The restriction $\pi_\alpha = \pi|_{G^p_\alpha(X)}$ is a fibre map of $G^p_\alpha(X)$ onto $X$. The homotopy properties of the various components $G^p_\alpha(X)$ of $G^p(X)$ have been studied by M. Abe (Jap. J. Math. vol. 16 (1940) pp. 169–176), G. W. Whitehead [10] and S. T. Hu [2]. The present note may be regarded as a continuation of these studies.

2. $H$-space and $H_\ast$-space. In what follows, we shall denote $G^p(X)$ by $G^p$ and $F^p(X, x)$ by $F^p$ whenever no confusion is likely to arise.

Let $X$ be a topological space which admits a continuous multiplication $\mu(x, x') = x \cdot x'$. If $f: S \to X$ is a map of a space $S$ into $X$, we denote by $x \cdot f$ the transformation defined by $(x \cdot f)(s) = x \cdot f(s)$ for each $s \in S$. Clearly $x \cdot f$ is a map (i.e. it is continuous).

By an $H$-space we mean a topological space $X$ with a given continuous multiplication which has a homotopy unit $e \in X$ (see e.g. [3, pp. 80–81]).

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Theorem. If $X$ is an arcwise connected $H$-space, then $G^p_\alpha(X)$ and $G^p_\beta(X)$ have the same homotopy type for arbitrary $\alpha$ and $\beta$ in $\pi_p(X)$, $p \geq 1$.

Proof. It suffices to prove that $G^p_\alpha(X)$ and $G^p_\beta(X)$ have the same homotopy type, for any $\alpha \in \pi_p(X)$. According to [10], it remains to prove that $G^p_\alpha(X)$ admits a (global) cross-section. Choose an element $f \in G^p_\alpha \cap F^p(X, e)$. Then $\pi_\alpha(f) = e$. Define $\phi : X \to G^p_\beta$ by $\phi(x) = x \cdot f$. Then $\phi(e) = e \cdot f \in G^p_\alpha$. Since $X$ is arcwise connected, we have $\phi : X \to G^p_\beta$.

Now, $\pi_\alpha(\phi(x)) = \pi_\alpha(x \cdot f) = x \cdot f_\alpha$, therefore $\pi_\alpha \phi \simeq \text{id}_X$. Since $\pi_\alpha$ has the absolute covering homotopy property, there exists a covering homotopy, in particular, there is a map $\psi : X \to G^p_\alpha$ such that $\pi_\alpha \psi = \text{id}_X$. This proves (2.1).

Following H. Wada [9], we call a topological space $X$ an $H_\ast$-space if the following conditions are satisfied:

(i) A continuous multiplication $\mu(x, x') = x \cdot x'$ is defined for each pair of elements $x, x'$ in $X$.

(ii) There is a fixed element $e$ in $X$, satisfying

$$x \cdot e = x,$$

for all $x \in X$.

(iii) To each $x \in X$, there is an inverse $x^{-1} \in X$, defined continuously by $x$, such that

$$x \cdot x^{-1} = e,$$

for all $x \in X$.

(iv) For each pair of elements $x, x'$ in $X$, we have

$$x^{-1} \cdot (x \cdot x') = x'.$$

With these conditions Wada was able to prove that

(ii') $e$ is unique,

(iii') $x^{-1}$ is uniquely defined by $x$ and $x^{-1} \cdot x = e$,

(v) $(x^{-1})^{-1} = x$ and, consequently, $x \cdot (x^{-1} \cdot x') = x'$, for arbitrary $x$ and $x'$ in $X$.

We remark that an $H_\ast$-space need not to be an $H$-space.

The following theorem resembles a construction of Wada [9], where he deals with mapping space of an $H_\ast$-space into itself.

Theorem. Let $X$ be an $H_\ast$-space. The mapping space $G^p(X)$ is homeomorphic to $X \times F^p(X, e)$ for each $p \geq 1$.

Proof. Let $g \in G^p(X)$ be an arbitrary map of $S^p$ into $X$. Then $x \cdot g$ is defined and continuous. Hence $x \cdot g \in G^p(X)$. Clearly $g = e \cdot g = x^{-1} \cdot (x \cdot g) = x \cdot (x^{-1} \cdot g)$ for any $x \in X$. Let
\[ \phi: G^p(X) \rightarrow X \times F^p(X, \epsilon), \]

and

\[ \psi: X \times F^p(X, \epsilon) \rightarrow G^p(X), \]

be defined as follows: Let \( y_0 \) be the pole of \( S^p \). For each \( g \in G^p(X) \), let \( \hat{g} = g(y_0) \in X \). Then define

\[ \phi(g) = (\hat{g}, \hat{g}^{-1} \cdot g), \quad (g \in G^p(X)) \]

and

\[ \psi(x, f) = x \cdot f, \quad (x \in X, f \in F^p(X, \epsilon)). \]

(A) \( \phi \) and \( \psi \) are bijective:

For any \( g \in G^p(X) \), we have

\[ \psi \phi(g) = \psi(\hat{g}, \hat{g}^{-1} \cdot g) = \hat{g} \cdot (\hat{g}^{-1} \cdot g) = g. \]

On the other hand,

\[ \phi \psi(x, f) = \phi(x \cdot f) = ((x \cdot f)^\epsilon, ((x \cdot f)^\epsilon)^{-1} \cdot (x \cdot f)) \]

\[ = (x \cdot f, (x \cdot f)^{-1} \cdot (x \cdot f)) \]

\[ = (x, x^{-1} \cdot (x, f)) \]

\[ = (x, f). \]

Hence both \( \phi \) and \( \psi \) are one-to-one, onto.

(B) \( \phi \) and \( \psi \) are continuous:

Suppose \( K \) be a compact set in \( S^p \) and \( U \) an open set in \( X \). We shall denote by \( (K, U) \) the subset of \( G^p(X) \) consisting of all mappings which send \( K \) into \( U \). Let \( H \) be an arbitrary neighborhood of \( (\hat{g}, \hat{g}^{-1} \cdot g) \). Then \( H \supseteq \bigcap_{i=1}^n (K_i, U_i) \) for some open sets \( U_0, U_1, \ldots, U_n \) in \( X \) and compact sets \( K_1, \ldots, K_n \) in \( S^p \). Denote \( g(K_i) \) by \( K'_i \), then \( K'_i \) is compact, \( i = 1, 2, \ldots, n \). Corresponding to each \( k'_i \in K'_i \), there exist open sets \( W_i^x \) containing \( k'_i \) and \( V_i^x \) containing \( k_i^x \) such that \( W_i^x \cdot V_i^x \subseteq U_i \), since the multiplication in \( X \) is continuous. The collection \( \{ V_i^x \} \) forms an open covering of \( K'_i \). There is a finite subcovering \( \{ V_1^x, \ldots, V_m^x \} \) of \( K'_i \). Let \( W_i = \bigcap_{j=1}^m W_i^x \) and \( V_i = \bigcup_{j=1}^m V_i^x \). Then \( W_i \) is an open neighborhood of \( k'_i \); \( V_i \) is an open neighborhood of \( K'_i \) and \( W_i \subseteq U_i \).

Let \( N = (y_0, U_0 \cap W_1^{-1} \cap \cdots \cap W_n^{-1}) \cap (K_1, V_1) \cap \cdots \cap (K_n, V_n) \), where \( W_i^{-1} \) denotes, of course, the set \( \{ w^{-1} | w \in W_i \} \). By the continuity of the inverse, \( N \) is a neighborhood of \( g \) in \( G^p \). It is now readily seen that \( \phi(N) \subseteq H \). This proves the continuity of \( \phi \).

Next, let \( U = (K_1, U_1) \cap (K_2, U_2) \cap \cdots \cap (K_n, U_n) \) be a basic open
neighborhood of $\psi(x, f) = x \cdot f$. Then $x \cdot f(K_i) \subset U_i$. By a similar argument as above, one proves that there exist open neighborhoods $W_i$ of $x$ and $V_i$ of $f(K_i)$ such that $W_i \cdot V_i \subset U_i$. Then

$$\psi[(W_1 \cap \cdots \cap W_n) \times ((K_1, V_1) \cap \cdots \cap (K_n, V_n) \cap F^p)] \subset U.$$ 

Hence $\psi$ is continuous and the proof of (2.2) is completed.

(2.3) **Corollary.** If $X$ is an arcwise connected $H_*$-space, then $G^p_\alpha$ and $X \times F^p_\alpha$ are homeomorphic.

**Proof.** Since $X$ is arcwise connected, $G^p_\alpha$ is a fibre space over $X$. By replacing $G^p$ and $G^p_\alpha$ and $\pi$ by $\pi_\alpha$ in the proof of (2.2), we obtain that $G^p_\alpha$ is homeomorphic to $X \times \pi_\alpha^{-1}(e)$. Being a component, $G^p_\alpha$ is connected hence $\pi_\alpha^{-1}(e)$ contains only one component $F^p_\alpha$. This proves (2.3).

As a by-product of the proof of (2.3) we have:

(2.4) **Corollary.** Every arcwise connected $H_*$-space is $n$-simple, for $n \geq 1$.

(2.5) **Corollary.** If $X$ is an arcwise connected $H_*$-space, then $G^p_\alpha$ and $G^p_\beta$ have the same homotopy type for arbitrary $\alpha$ and $\beta$ in $\pi_p(X)$. Furthermore

$$\pi_q(G^p_\alpha) \approx \pi_{p+q}(X) + \pi_q(X), \quad (q \geq 1).$$

**Proof.** Since G. W. Whitehead [10] proved that $F^p_\alpha$ and $F^p_\beta$ have the same homotopy type for any $\alpha$ and $\beta$ in $\pi_p(X)$, the first part of (2.5) follows from (2.3). The Hurewicz isomorphism $\pi_q(F^p_\alpha) \approx \pi_{p+q}(X)$ (cf. [10]) completes the proof.

(2.6) **Corollary.** Let $X = S^r$. Then $G^p_\alpha$ is homeomorphic to $S^r \times F^p_\alpha$ when $r = 1, 3$ or 7. Conversely, if $G^p_\alpha$ and $S^r \times F^p_\alpha$ have the same homotopy type then $r = 1, 3$ or 7, where $i_r \subset \pi_r(S^r)$ is represented by the identity map $S^r \rightarrow S^r$.

**Proof.** This follows from Wada [8] and a recent result of Adams [1].

(2.6) **Proposition.** If $X$ is a $H$-space, then for each $\alpha \in \pi_p(X)$,

$$\pi_q(G^p_\alpha) / \pi_{p+q}(X) \approx \pi_q(X),$$

where $\pi_{p+q}(X)$ is, of course, imbedded in $\pi_q(G^p_\alpha)$ isomorphically.

**Proof.** According to G. W. Whitehead [10] (see also [11]), we have the following diagram:
3. The sphere \( S^r \). Let \( X = S^r \), an \( r \)-sphere, then we have the following exact sequence

\[
\cdots \to \pi_0(F^p_a) \xrightarrow{i^*} \pi_0(G^p_a) \xrightarrow{j^*} \pi_0(G^p_{a, \alpha}) \xrightarrow{\partial} \pi_{-1}(F^p_a) \to \cdots
\]

where \( \pi_* \) denotes the isomorphism induced by the projection \( \pi \), \( H \) denotes the Hurewicz isomorphism and \( \rho_{\alpha} \) is defined by \( \rho_{\alpha}(\beta) = -[\alpha, \beta] \). Since \( \rho_{\alpha} \) is always trivial when \( X \) is an \( H \)-space, (2.7) follows from the exactness of the sequence.

The following propositions are fairly obvious.

(3.2) PROPOSITION. Let \( X = S^r \) and \( \alpha \in \pi_p(S^r) \). Since \( \pi_q(S^r) = 0 \) for \( q < r \) we have

\[
\pi_q(G^p_a) \approx \pi_{p+q}(S^r), \quad (q < r - 1).
\]

(3.3) COROLLARY. \( \pi_1(G^p_a) \approx \mathbb{Z}_2 \) for \( r \geq 3 \).

Since \( \pi_{r+2}(S^r) \approx \mathbb{Z}_2 \) for \( r \geq 3 \), we have

(3.4) COROLLARY. \( \pi_1(G^{r+1}_a) \approx \mathbb{Z}_2 \), \( r \geq 3 \).

Denote the image of \( \rho_{\alpha}: \pi_q(S^r) \to \pi_{p+q-1}(S^r) \) by \( J^{p+q-1}_{a, \alpha} \) and the kernel of \( \rho_{\alpha} \) by \( K^p_a \). Denote the image of \( \mu: \pi_{p+q}(S^r) \to \pi_q(G^p_a) \) by \( P_a \). Then

(3.5) PROPOSITION (Hu) [2]. For \( X = S^r \) and \( \alpha \in \pi_p(S^r) \)

(a) \( \pi_q(G^p_a)/P^p_a \approx K^q_a \), \( (q > 1) \),

(b) \( \pi_{p+q}(G^p_a)/J^{p+q}_{a, \alpha} \approx P^q_a \), \( (q > 1) \),

(c) \( \pi_{p+r-1}(G^p_a) \approx \pi_{p+r-1}(S^r)/J^{p+r}_{a, \alpha} \),

(d) \( \pi_{p+3}(G^p_a) \) has a subgroup \( P^{p+3}_{a, \alpha} \approx \pi_{p+r+3}(S^r) \), \( (r \geq 6) \),

(e) \( \pi_{p+4}(G^p_a) \approx \pi_{p+r+4}(S^r)/J^{p+r+4}_{a, \alpha} \), \( (r \geq 6) \).

Since for \( r \geq 7 \), \( \pi_{r+4}(S^r) = \pi_{r+5}(S^r) = 0 \). It follows that

(3.6) PROPOSITION. If \( r > 7 \), for each \( \alpha \in \pi_p(S^r) \),

\[
\pi_{r-1}(G^p_a) \approx \pi_{r-2}(G^p_a) \approx \cdots \approx 0.
\]

And,
(3.7) Proposition. For $r \geq 7$, $\alpha \in \pi_p(S^r)$, 
\[ \pi_{r+8-p}(G^p) \approx \pi_{r+8-p}(S^r). \]

We now proceed to prove the main theorem of this section. Consider the following sequence 
\[ (\ref{eq:sequence}) \quad \pi_r(S^r) \xrightarrow{\rho_\alpha} \pi_{2r-1}(S^r) \xrightarrow{E} \pi_{2r}(S^{r+1}), \]
where $E$ denotes the Freudenthal suspension. By the delicate suspension theorem, the kernel of $E$ is a cyclic subgroup generated by $[i_r, \iota_r]$. If $r$ is even, it is infinite cyclic; if $r$ is odd, it is cyclic of order 2.

(3.8) Lemma (Hu). For $X = S^2$ and $\alpha \in \pi_2(S^2)$, we have 
\[ \pi_1(G^2) \approx \mathbb{Z}_{2m}, \]
where $m$ is the absolute value of the degree of $\alpha$.

**Proof.** Since $\pi_{2r}(S^{r+1}) = \pi_4(S^8) \approx \mathbb{Z}_2$. From (3.8) $\pi_8(S^8)/\ker E \approx \mathbb{Z}_2$. Let $\gamma$ be a generator of the free cyclic group $\pi_3(S^2)$. Then $[i_2, \iota_2] = \pm 2$. We can choose $\gamma$ so that $[i_2, \iota_2] = -2\gamma$. Let $\alpha \in \pi_2(S^2)$. By linearity of the Whitehead product $\rho_\alpha(i_2) = -[\alpha, \iota_2] = -m[i_2, \iota_2] = 2m\gamma$. In other words $J^2_\alpha$ is generated by $2m\gamma$. From (3.5(c)), we have $\pi_1(G^2) \approx \mathbb{Z}_{2m}$. This proves (3.9).

(3.9) Lemma. For $X = S^4$ and $\alpha \in \pi_4(S^4)$, we have 
\[ \pi_3(G^4) \approx \mathbb{Z}_{24m} + \mathbb{Z}_{12}, \]
where $m$ is the absolute value of the degree of $\alpha$.

**Proof.** Since $\pi_{2r}(S^{r+1}) = \pi_8(S^8) \approx \mathbb{Z}_2$ and $\pi_{2r-1}(S^r) = \pi_7(S^4) \approx \mathbb{Z} + \mathbb{Z}_{12}$. One generator of $\ker E$ is determined as follows:

From a theorem of characteristic map [5, p. 121], that 
\[ [\iota_4, \iota_4] = 2[q] - \epsilon E[\xi], \]
where $\epsilon = \pm 1$ depends on the convention of orientation, $[q]$ denotes the homotopy class of the Hopf map $q: S^7 \to S^4$ and $[\xi]$ a generator of $\pi_6(S^6)$ represented by the characteristic map $\xi: S^6 \to S^8$ of the fibre bundle $Sp(2)$ over $S^7$ with $Sp(1)$ as fibre. Hence in $\pi_8(S^8)$ we have 
\[ E^2[\xi] = \epsilon 2E[q], \]
($E^2$ denotes the iterated suspension). This implies that $\pi_8(S^6)$ has $E[q]$ as a generator. Hence 
\[ \pi_7(S^4)/\ker E \approx \mathbb{Z}_{24} + \mathbb{Z}_{12}. \]
A similar argument as used in (3.9) yields

\[ \pi_3(G_\alpha^4) \approx Z_{24m} + Z_{12}. \]

(3.11) **Lemma.** For \( X = S^6 \) and \( \alpha \in \pi_6(S^6) \), we have

\[ \pi_6(G_\alpha^6) \approx Z_m, \]

where \( m \) is the absolute value of the degree of \( \alpha \).

**Proof.** Since \( \pi_{2r}(S^{r+1}) = \pi_{12}(S^7) = 0 \) and \( \pi_{2r-1}(S^6) = \pi_{11}(S^6) \approx Z \), \( \text{Ker } E = J^1 \). Hence we can choose the generator \( \gamma \) of \( \pi_{11}(S^6) \) such that \( \gamma = -[\iota_6, \iota_6] \), consequently \( \rho_\alpha(\iota_6) = m\gamma \), or \( \pi_6(G_\alpha^6) \approx Z_m \) by (3.4(c)).

(3.12) **Lemma.** For \( X = S^8 \) and \( \alpha \in \pi_8(S^8) \), we have

\[ \pi_7(G^8) \approx Z_{240m} + Z_{120}, \]

where \( m \) is the absolute value of the degree of \( \alpha \).

**Proof.** \( \pi_{2r}(S^{r+1}) = \pi_{16}(S^9) \approx Z_{240} \) and \( \pi_{2r-1}(S^8) = \pi_{15}(S^8) \approx Z + Z_{120} \). Since \( [\iota_6, \iota_6] = 2[q'] - \epsilon E[\xi'] \), where \( [q'] \) denote the homotopy class represented by the Hopf map \( q' : S^{16} \to S^8 \) and \( \xi' \in \pi_{14}(S^7) \) has nonzero Hopf invariant, we have

\[ E^2[\xi'] = 2\epsilon E[q']. \]

Using the same argument as in (3.10), one proves (3.12).

(3.13) **Lemma.** For \( X = S^{10} \) and \( \alpha \in \pi_{10}(S^{10}) \), we have

\[ \pi_9(G^{10}) \approx Z_m + Z_2 + Z_2 + Z_2, \]

where \( m \) denotes the absolute value of the degree of \( \alpha \).

(3.14) **Lemma.** For \( X = S^{12} \) and \( \alpha \in \pi_{12}(S^{12}) \), we have

\[ \pi_{11}(G^{12}) \approx Z_m + Z_8 + Z_7 + Z_7, \]

where \( m \) denotes the absolute value of the degree of \( \alpha \).

The proof of (3.13) follows from the table in Toda [6] the first row and a similar argument as before; for a proof of (3.14), one uses the third row of the above mentioned table.

(3.15) **Lemma.** For \( X = S^{14} \) and \( \alpha \in \pi_{14}(S^{14}) \), we have

\[ Z_{13}(G^{14}) \approx Z_m + Z_3, \]

where \( m \) denotes the absolute value of the degree of \( \alpha \).

**Proof.** Since \( \pi_{17}(S^{14}) \approx Z + Z_3 \) and \( \pi_{18}(S^{16}) \approx Z_3 \) and the suspension \( E \) sends \( Z \) into 0 in \( \pi_{18}(S^{16}) \) (Toda [6]). The proof is immediate.
(3.16) **Lemma.** For \( X = S^r \), \( \alpha \in \pi_r(S^r) \) and \( r \) odd, \( \neq 1, 3, 7 \). Then

(a) \( \pi_{r-1}(G^r) \approx \pi_{2r-1}(S^r) \) when \( \alpha \) is of even degree,

(b) \( \pi_{r-1}(G^r) \approx \pi_{2r-1}(S^r)/\mathbb{Z}_2 \) when \( \alpha \) is of odd degree.

**Proof.** It suffices to prove that there is a nonzero element in \( J_{\alpha}^{2r-1} \) when \( \alpha \) is of odd degree and \( r \neq 1, 3, 7 \). In fact, in this case \( \rho_\alpha (\nu_r) \neq 0 \). (3.16) follows.

(3.17) **Lemma (Hu).** Let \( X \) be any space. If \( \alpha, \beta \in \pi_p(X) \), \( \alpha + \beta = 0 \). Then \( G^p_\alpha \) and \( G^p_\beta \) are homeomorphic.

**Proof.** Let \( \theta : S^p \to S^p \) be a homeomorphism which reverses the orientation and leaves the pole \( y_0 \) fixed. Then a homeomorphism \( h \) of \( G^p_\alpha \) onto \( G^p_\beta \) is given by \( h(f) = f \circ \theta \) for each \( f \in G^p_\alpha \).

(3.18) **Theorem.** Let \( X = S^r \). Let \( \alpha, \beta \in \pi_r(S^r) \). Then for \( r = 2, 4, 6, 8, 10, 12, 14 \), the components \( G^r_\alpha \) and \( G^r_\beta \) have the same homotopy type if and only if \( \alpha = \pm \beta \). When \( r \) is odd \( \neq 1, 3, 7 \), the components \( G^r_\alpha \) and \( G^r_\beta \) are of different homotopy type if \( \deg \alpha - \deg \beta \) is odd.

**Proof.** The first part of the theorem follows from Lemmas (3.9) through (3.17). The remaining part follows from the fact that if \( r \) is odd then \( \pi_p(S^r) \) is finite for \( p > n \) [4].

(3.19) **Corollary.** Let \( X = S^r \) and \( \alpha, \beta \in \pi_r(S^r) \) are of odd and even degree respectively. Then:

\[
\begin{align*}
\pi_4(G^8_\alpha) &= 0, & \pi_4(G^8_\beta) &\approx \mathbb{Z}_2, \\
\pi_8(G^8_\alpha) &= Z_2 + Z_2, & \pi_8(G^8_\beta) &\approx Z_2 + Z_2 + Z_2, \\
\pi_{10}(G^8_\alpha) &= Z_2 + Z_9, & \pi_{10}(G^8_\beta) &\approx Z_2 + Z_2 + Z_9, \\
\pi_{12}(G^8_\alpha) &= 0, & \pi_{12}(G^8_\beta) &\approx Z_2, \\
\pi_{14}(G^8_\alpha) &= Z_2 + Z_2 \text{ or } Z_4, & \pi_{14}(G^8_\beta) &\approx Z_4 + Z_2.
\end{align*}
\]

(3.20) **Proposition (Hu).** When \( r \) is even and \( \alpha \in \pi_r(S^r) \), \( r \neq 0 \). Then

\[
\pi_r(G^r_\alpha) \approx \pi_{2r}(S^r)/J_{\alpha}^{2r}.
\]

**Proof.** Since \( K^r_\alpha = 0 \), the result follows from (3.4(a)) and (3.4(b)).

(3.21) **Proposition.** If \( E : \pi_p(S^r) \to \pi_{p+1}(S^{r+1}) \) is an injection, then for \( q + s = p \) and \( q > 1 \)

\[
\pi_q(G^r_\alpha)/\pi_p(S^r) \approx \pi_q(S^r),
\]

where \( \pi_p(S^r) \) is imbedded in \( \pi_q(G^r_\alpha) \).
Proof. Since \( E[\alpha, \beta] = 0 \), \( J^p_\alpha \subset \text{Ker } E = 0 \). From (3.4)(a) and (b), \( \pi_q(G^*_\alpha)/\pi_{q+s}(S^r) \approx K^*_\alpha \). But \( K^*_\alpha = \pi_q(S^r) \). This proves (3.21).

For \( q < r \), \( \pi_q(S^r) = 0 \), we have \( \pi_q(G^*_\alpha) \approx \pi_p(S^r) \). This reduces to (3.2).

(3.22) Corollary. If \( q + s = \rho < 2r - 1 \), then

\[
\pi_q(G^*_\alpha)/\pi_p(S^r) \approx \pi_q(S^r).
\]

Proof. This follows from (3.21).

Bibliography


Wayne State University