ON THE POLYHEDRAL SCHÖNFIßES THEOREM

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In this note we observe a relationship between the polyhedral Schönfliess problem and the question of whether the double suspension $M^6$ of a Poincaré manifold is the 5-sphere. In particular we show that if $M^6 = S^6$, then a polyhedral embedding of $S^{n-1}$ in $S^n$ must be very "nice" if the Schönfliess theorem is to hold.

**Definition 1.** Let $\Delta^n$ be the standard $n$-simplex and $\Delta^n$ be its boundary. A finite simplicial complex $K$ is a *combinatorial $n$-sphere* if there exists a piecewise linear homeomorphism $h: K \to \Delta^{n+1}$.

**Definition 2.** An embedding $S^{n-1} \subset S^n$ is *nice* if there is a simplicial decomposition of $S^n$ such that $S^{n-1}$ is a subcomplex and both $S^{n-1}$ and $S^n$ are combinatorial spheres.

We have the following theorem (see [5]):

**Theorem 1.** If the embedding $S^{n-1} \subset S^n$ is nice, then the Schönfliess theorem holds; i.e., $S^n - S^{n-1}$ consists of two disjoint $n$-cells.

**Definition 3.** An embedding $S^{n-1} \subset S^n$ is of *type I* if $S^n$ can be represented as a combinatorial $n$-sphere with $S^{n-1}$ a subcomplex. An embedding is of *type II* if there is a simplicial decomposition of $S^n$ such that $S^{n-1}$ is a subcomplex which is a combinatorial $(n-1)$-sphere.

We construct a definite Poincaré manifold $M^3$ in $S^4$. Let $P$ be the 2-polyhedron obtained by attaching the boundaries of two disks to two oriented curves $a$ and $b$ (with one common point) according to the formulae $a^{-2}bab = 1$, $b^{-2}abab = 1$. Then $\pi_1(P)$ has the presentation \{a, b \, | \, a^{-2}bae = 1, b^{-2}abab = 1\}, and Newman [4] has shown that $\pi_1(P) \neq 0$. Now $P$ can be embedded in $S^4$ as a subcomplex (see [2; 4]) with $S^4$ decomposed as a combinatorial 4-sphere. Then the boundary $M^3$ of a nice neighborhood of $P$ is a Poincaré manifold [2]. It follows that the double suspension $M^6$ of $M^3$ is a subcomplex of the combinatorial 6-sphere $S^6$. This is used in Theorem 3.

We note that if $M^3$ is locally euclidean, then $M^6 = S^6$. For Mazur [3] has proved that if $X$ is a finite polyhedron and the cone $C(X)$ is locally $k$-euclidean at the cone point, then $C(X) - X = E^k$. Now $M^6$ is the suspension of the single suspension $M^4$ of $M^3$ and the suspension with one suspension point removed is just $C(M^4) - M^4$. If this

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is $E^8$, then the suspension is just the 1-point compactification of $E^8$, namely $S^8$.

**Theorem 2.** If $M^5 = S^5$, then the Schoenflies theorem fails for embeddings of type II with $n = 5$.

**Proof.** Let $\sigma$ be a 3-simplex of $M^5$ with boundary $\beta$. Then the double suspension of $\beta$ is a combinatorial 4-sphere $S^4$ in $M^5 = S^8$. But $\pi_1(M^5 - \sigma) = \pi_1(M^5) \neq 0$ and one complementary domain of $S^4$ in $S^8$ is just $(M^5 - \sigma) \times I$, which is not simply connected.

**Theorem 3.** If $M^5 = S^5$, then the Schoenflies theorem fails for embeddings of type I with $n = 6$.

**Proof.** By the construction of $M^3 \subset S^4$ given above, we have that $M^5 = S^5 \subset S^8$ is an embedding of type I with $n = 6$. Let $D^5$ be the complementary domain of $M^3$ in $S^4$ which contains $P$. By projecting from suspension points we can get deformation retraction of a complementary domain $D^5$ (of $S^5 \subset S^8$) onto $D^4$ (of $M^4 \subset S^8$) onto $D^3$. Hence $D^6$ is not simply connected and therefore is not a cell.

**Remark.** Since it seems difficult to prove that $M^5 \neq S^5$, it must be hard to show that embeddings of types I and II satisfy the simple condition $S^{n-1} \times I \subset S^n$ which Morton Brown [1] has shown is a necessary and sufficient hypothesis for the Schoenflies theorem.

**References**


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