ABSOLUTE-VALUED ALGEBRAS

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An algebra A over the real field R is a vector space over R which is closed with respect to a product xy which is linear in both x and y, and which satisfies the condition $\lambda(xy)=(\lambda x)y=x(\lambda y)$ for any λ in R and x, y in A. The product is not necessarily associative. An element e of the algebra A is called a unit element if $ex=xe=x$ for any x in A. Given any subset B of A, dim B will denote the linear dimension of B; i.e., the power of a maximal set of linearly independent elements of B. Further, $[B]$ will denote the linear set spanned by the elements of B. For each x in A, we shall denote by $A(x)$ the subalgebra generated by x. The algebra A is called algebraic if $A(x)$ is finite dimensional for every x in A. The algebra A is said to be a division algebra if for every a, b in A, with $a\neq 0$, the equations $ax=b$ and $ya=b$ are solvable in A.

An algebra over R is called absolute-valued if it is a normed space under a multiplicative norm $|\ |$; i.e., a norm satisfying, in addition to the usual requirements, the condition $|xy|=|x|\cdot|y|$ for any x, y in A. It is obvious that an absolute-valued algebra contains no divisors of zero.

A. A. Albert has shown [2, p. 768] that:

(*) An absolute-valued algebraic algebra with a unit element is isomorphic to either the real field R, the complex field C, the quaternion algebra Q, or the Cayley-Dickson algebra D.

F. B. Wright has proved [6, p. 332] the same theorem for absolute-valued division algebras with a unit element. In the present note we extend this result to an arbitrary absolute-valued algebra with a unit element.

First, we shall give a simple example of an infinite dimensional algebra which is absolute-valued. The existence of such an algebra shows that the assumption of the presence of a unit element is essential.

Let A_0 be the space of all sequences $x=\{x_n\}$ of real numbers with convergent series $\sum_{n=1}^{\infty} x_n^2$. A_0 is a Hilbert space over R with respect to the norm $|x|=(\sum_{n=1}^{\infty} x_n^2)^{1/2}$, and with the usual addition and scalar multiplication: $\{x_n\}+\{y_n\}=\{x_n+y_n\}$, $\lambda\{x_n\}=\{\lambda x_n\}$. Let ϕ

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be a one-to-one correspondence of the set of all pairs of natural numbers onto the set of all natural numbers. We define the multiplication of elements in A_0 as follows: $\{x_n\}\{y_n\} = \{z_n\}$, where $z_{\phi(m,k)} = x_m y_k$ $(m, k=1, 2, \cdots)$. This product makes A_0 an algebra over R. Moreover, A_0 is absolute-valued. Indeed, we have the equality

$$\begin{aligned} |xy| &= \left(\sum_{n=1}^{\infty} z_n^2\right)^{1/2} = \left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} z_{\phi(m,k)}^2\right)^{1/2} \\ &= \left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} x_m^2 y_k^2\right)^{1/2} = \left(\sum_{m=1}^{\infty} x_m^2\right)^{1/2}\left(\sum_{k=1}^{\infty} y_k^2\right)^{1/2} \\ &= |x|\,|y|. \end{aligned}$$

Since A_0 is a Hilbert space, the function $|x|^2$ from A_0 to R is a quadratic form admitting composition with respect to this multiplication. The complete structure theory of algebras (over any field) admitting such a form has been given by Kaplansky [**4**, p. 957], under the hypothesis that the algebra has a unit element. The algebra A_0 above shows that this hypothesis of existence of a unit element is essential to Kaplansky's results.

Throughout this paper, A will denote an absolute-valued algebra.

LEMMA 1. *If all the elements of a subset B of A commute with each other, then $[B]$ is an inner-product space.*

PROOF. For every pair x, y of elements of B we have $(x+y)^2 - (x-y)^2 = 4xy$. Consequently, for $|x| = |y| = 1$, we get the inequality

$$|x + y|^2 + |x - y|^2 = |(x + y)^2| + |(x - y)^2| \geqq 4|x|\cdot|y|.$$

Hence, according to Schoenberg's theorem [**5**, p. 962], we know that $[B]$ is an inner-product space over R.

LEMMA 2. *Let x and y be a pair of linearly independent elements of A. If x commutes with y and if $|x| = 1$, then there exists an element y_0 such that $[x, y_0] = [x, y]$ and*

$$|\lambda x + \mu y_0|^2 = \lambda^2 + \mu^2 \tag{1}$$

for any λ, μ in R.

PROOF. By Lemma 1, $[x, y]$ is an inner-product space. Since x and y are linearly independent, $[x, y]$ is a two-dimensional linear space. There is then an element y_0 with $|y_0| = 1$ such that y_0 is orthogonal to x and such that $[x, y] = [x, y_0]$. Equation (1) is a direct consequence of the orthogonality of x and y_0.

LEMMA 3. *Let* $x, y \in A$, $|x| = |y| = 1$, *and* $|x-y| = 2$. *If* x *commutes with* y, *then* $x+y=0$.

PROOF. If $x=\lambda y$, $\lambda \in R$, we have the equalities $|\lambda| = |\lambda y| = |x| = 1$, $|\lambda - 1| = |\lambda y - y| = |x-y| = 2$, which imply $\lambda = -1$. Thus $x+y=0$. Suppose then that x and y are linearly independent and commute with each other. Then, by Lemma 2, there exists an element y_0 such that $[x, y] = [x, y_0]$ and

$$|\lambda x + \mu y_0|^2 = \lambda^2 + \mu^2. \tag{2}$$

The element y can be written in the form

$$y = \alpha x + \beta y_0, \tag{3}$$

where, by (2), we have

$$1 = |y|^2 = \alpha^2 + \beta^2. \tag{4}$$

Furthermore, we have the equality

$$4 = |x - y|^2 = |(1 - \alpha)x + \beta y_0|^2 = (1 - \alpha)^2 + \beta^2.$$

From (4) we have $\alpha = -1$ and $\beta = 0$. Substituting in (3) yields $x+y=0$.

THEOREM 1. *An absolute-valued algebra with a unit element is isomorphic to either the real field* R, *the complex field* C, *the quaternion algebra* Q, *or the Cayley-Dickson algebra* D.

PROOF. By virtue of Albert's theorem (*), it is sufficient to show that every absolute-valued algebra A with a unit element e is algebraic. We will show that

$$x^2 \in [e, x] \tag{5}$$

for any x in A, which is equivalent to the inclusion $A(x) \subset [e, x]$, and which consequently implies the inequality $\dim A(x) \leqq 2$.

If e and x are linearly dependent, then (5) is obvious. Let us suppose that they are linearly independent. By Lemma 2, there exists an element x_0 such that

$$[e, x] = [e, x_0], \tag{6}$$

and such that

$$|\lambda e + \mu x_0|^2 = \lambda^2 + \mu^2 \qquad (\lambda, \mu \in R). \tag{7}$$

From (6) follows the relation

$$x^2 \in [e, x_0, x_0^2]. \tag{8}$$

Further, from (7) we have the equality

$$|e - x_0^2| = |(e - x_0)(e + x_0)| = |e - x_0| \cdot |e + x_0| = 2.$$

Since e commutes with x_0 and since $|e| = 1 = |x_0| = |x_0^2|$, Lemma 3 asserts that $e + x_0^2 = 0$. Thus $[e, x_0, x_0^2] = [e, x_0]$. This, together with (6) and (8), gives relation (5). The theorem is thus proved.

The real field R is the unique one-dimensional absolute-valued algebra. The structure of two-dimensional absolute-valued algebras is also well-known. In particular, every two-dimensional commutative absolute-valued algebra is isomorphic to either the complex field C or to the algebra C^* of all complex numbers with the usual addition and scalar multiplication, where the product of x and y is equal to $\bar{x}\bar{y}$ [**1**; **3**].

THEOREM 2. *If an absolute-valued algebra A contains an element $a \neq 0$ which commutes with every element of A and which is alternative, i.e., which satisfies the equations*

$$a(ax) = a^2x, \qquad (xa)a = xa^2, \tag{9}$$

for any x in A, then A has a unit element.

PROOF. We may suppose, without loss of generality, that $|a| = 1$. If $a^2 = \lambda a$, where λ is in R, we may set $e = \lambda^{-1}a$, and we have

$$e^2 = \lambda^{-2}a^2 = \lambda^{-1}a = e,$$
$$e(ex) = \lambda^{-2}a(ax) = \lambda^{-2}a^2x = e^2x = ex.$$

Hence we have the equation

$$e(ex - x) = e(ex) - ex = 0,$$

which implies $ex - x = 0$ for any x in A. Since e commutes with every element of A, e is a unit element of A.

Now let us suppose that a and a^2 are linearly independent. By Lemma 2, there exists an element b such that

$$[a, a^2] = [a, b] \tag{10}$$

and such that $|\lambda a + \mu b|^2 = \lambda^2 + \mu^2$, for λ, μ in R. Hence we get

$$|a^2 - b^2| = |(a - b)(a + b)| = |a - b|\,|a + b| = 2, \tag{11}$$

$$|b^2| = |b|^2 = 1. \tag{12}$$

Since a commutes with every x in A, we have, according to (9), $a^2x = xa^2$ for any x in A. Therefore

$$xy = yx \quad \text{for } x \text{ in } A, \quad y \text{ in } [a, a^2]. \tag{13}$$

Taking into account (11), (12), and the relation $a^2b^2=b^2a^2$, we have, by virtue of Lemma 3, $a^2+b^2=0$. From (10) we get the representation $b=\alpha a+\beta a^2$, where $\beta\neq 0$ since a and a^2 are linearly independent. Hence $a^2+\alpha^2a^2+2\alpha\beta aa^2+\beta^2(a^2)^2=a^2+b^2=0$. From (9) we have the equality

$$a((1+\alpha^2)a+2\alpha\beta a^2+\beta^2 aa^2)=a^2+b^2=0,$$

which implies $(1+\alpha^2)a+2\alpha\beta a^2+\beta^2aa^2=0$. Thus aa^2 is in $[a, a^2]$. Writing $aa^2=\lambda a+\mu a^2$, we have, according to (9),

$$(a^2)^2=a(aa^2)=\lambda a^2+\mu aa^2\in[a, a^2].$$

Hence it follows that the product of an arbitrary pair of elements of $[a, a^2]$ also belongs to $[a, a^2]$. In other words, $[a, a^2]$ is a subalgebra of A, and $A(a)=[a, a^2]$. Hence $A(a)$ is a two-dimensional commutative absolute-valued algebra. Since the algebra C^* does not contain an element $a\neq 0$ satisfying (9), $A(a)$ is isomorphic to the complex field C. Then there are elements e_0, i_0 such that $A(a)=[e_0, i_0]$, $e_0^2=e_0$, $i_0^2=-e_0$, $e_0i_0=i_0e_0=i_0$.

To prove the theorem, it is sufficient to show that $A(a)=A$. Let us suppose to the contrary, that there exists an element x not in $A(a)$. Since, according to (13), e_0 and i_0 commute with x, then $[e_0, i_0, x]$ is an inner product space. There is then an element y in $[e_0, i_0, x]$, with $|y|=1$, orthogonal to both e_0 and i_0. Consequently

$$|y^2-e_0|=|(y-e_0)(y+e_0)|=|y-e_0|\,|y+e_0|=2,$$
$$|y^2+e_0|=|(y-i_0)(y+i_0)|=|y-i_0|\,|y+i_0|=2.$$

By Lemma 3, $y^2-e_0=0$ and $y^2+e_0=0$, which implies the contradiction $e_0=0$. The theorem is thus proved.

THEOREM 3. *A commutative absolute-valued algebra is isomorphic either to the real field R, or to the complex field C, or to the algebra C^*.*

PROOF. It is sufficient to show that every commutative absolute-valued algebra is at most two-dimensional. Suppose $\dim A\geqq 3$. By Lemma 1, A is an inner-product space. There exist orthonormal elements i_1, i_2, i_3 in A. From the equalities

$$|i_3^2-i_1^2|=|(i_3-i_1)(i_3+i_1)|=|i_3-i_1|\,|i_3+i_1|=2,$$

$$|i_3^2-i_2^2|=|(i_3-i_2)(i_3+i_2)|=|i_3-i_2|\,|i_3+i_2|=2,$$

and from Lemma 3, it follows that $i_3^2+i_1^2=0$ and $i_3^2+i_2^2=0$. Hence $i_1^2-i_2^2=(i_1+i_2)(i_1-i_2)=0$, which implies either $i_1+i_2=0$ or $i_1-i_2=0$.

Since i_1 and i_2 are orthonormal, either is a contradiction. Hence $\dim A \leqq 2$.

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