A NOTE ON SEMI-GROUPS IN A LOCALLY COMPACT GROUP

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1. **Introduction.** In a recent paper by this author and others [1], the following theorem is proved:

THEOREM 3 (SIMON). *In a compact group, every semi-group which contains a set of positive measure is an open and closed subgroup and therefore is itself measurable.*

In this paper, we show that this result can be improved to the following:

THEOREM A. *In a locally compact group, every semi-group of nonzero inner measure and finite outer measure is an open compact subgroup.*

This theorem can also be used to show an elusive[1] point in Theorem 5 of [1].

2. **Proof of Theorem** A. We rely heavily in this proof on Theorem 1 of [1], which states:

THEOREM 1. *Let G be a locally compact topological group with completed Haar measure μ and outer measure μ^*. Let A, $B \subset G$ be sets such that $\mu(A) > 0$ and $\mu^*(B) > 0$. Then the interior of BA (also AB) is nonempty.*

Let S now be a semi-group in a locally compact group G with Haar measure μ. We assume that $\mu_*(S) > 0$, so that S contains a measurable set of nonzero measure. Thus, by Theorem 1, the interior of $S^2 \subset S$ is nonempty. Hence S_0, the interior of S, is also nonempty. Since S has finite outer measure, S_0 is measurable of finite measure. It is also clear that $S_0 \cdot S \subset S_0$. For each $s \in S_0$, we have

$$s \cdot S_0 \subset S_0^2 \subset S_0 \cdot S \subset S_0,$$

so that, since μ is left invariant,

$$\mu(S_0) = \mu(s \cdot S_0) \leqq \mu(S_0^2) \leqq \mu(S_0).$$

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[1] The author expresses his debt to Mr. John E. Lange for pointing out this elusiveness.

Therefore

$$\mu(S_0 \backslash s \cdot S_0) = 0.$$

Since $s \cdot S_0 \subset s \cdot \bar{S}_0$, we see that $\mu(S_0 \backslash s \cdot \bar{S}_0) = 0$. But $S_0 \backslash s\bar{S}_0$ is open, so that it must be empty, and $s \cdot \bar{S}_0 \supset S_0$. We now set $U = s^{-1} \cdot S_0$. This is clearly a neighborhood of the identity in G, and we have $s \cdot \bar{S}_0 \supset S_0 \supset S_0 \cdot S_0 = s \cdot U \cdot S_0 \supset s \cdot \bar{S}_0$. Thus, $S_0 = s \cdot \bar{S}_0$, and S_0 is closed, so that $S_0 = s \cdot S_0$.

It follows immediately that S_0 is an open and closed subgroup of G. We now have

$$S = e \cdot S \subset S_0 \cdot S \subset S_0 \subset S.$$

Therefore $S = S_0$. Since S is an open and closed subgroup of finite measure, S is compact.

3. **Proof of part of Theorem 5 in [1].** Part of Theorem 5 of [1] requires the proof that if in a locally compact group G we have two measurable sets T and S with T a subgroup and S a sub-semigroup of G, and if $\mu(T) = \infty$ and $\mu(T \backslash S) < \infty$, then we have $T \subset S$. We construct the set $A = T \cap (S \backslash S^{-1})$; then we observe that A^{-1} is a semigroup, for if $a_1, a_2 \in A$, then $a_2 a_1 \in S$. However, $a_1^{-1} a_2^{-1} \notin S$, for that would give us $a_2^{-1} \in a_1 S \subset S$, and $a_2 \in S^{-1}$, contrary to assertion. Thus, $a_1^{-1} a_2^{-1} \in A^{-1}$, which is thus a semi-group. Since $A^{-1} = T \cap (S^{-1} \backslash S)$, we see that $\mu(A^{-1}) \leqq \mu(T \backslash S) < \infty$. If $\mu(A^{-1}) > 0$, we have, by Theorem A, that A^{-1} is an open subgroup, and thus contains e, which is false, since $A \cap A^{-1} = \square$. Thus $\mu(A) = 0$, and $T \cap S \cap S^{-1}$, which is a subgroup of T, has the additional property that

$$\mu(T \cap S \cap S^{-1}) = \infty,$$
$$\mu(T \backslash (S \cap S^{-1})) = \mu(T \backslash S) + \mu(A) < \infty.$$

It now follows easily (cf. [1, Lemma 5.1]) that $T = (S \cap S^{-1}) \cap T$, which gives us $T \subset S$.

Bibliography

1. Anatole Beck, H. H. Corson and A. B. Simon, *The interior points of the product of two subsets of a locally compact group*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 648–652.

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