SUMMABILITY OF A CLASS OF FOURIER SERIES

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1. In this section we shall consider a class of summability methods which sum the Fourier series

$$\frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1}$$

of a function $f(x)$ which is Lebesgue integrable and satisfies the condition

$$|f(x+h) - f(x)| = o\left[\left(\log \frac{1}{|h|}\right)^{-1}\right] \tag{2}$$

for some x. It is known that the Fourier series of such functions need not converge [3, p. 173]. The partial sums, $s_k(x)$, of (1) (since without loss of generality we may suppose $f(x)$ to be even) are given by:

$$s_k(x) = \frac{2}{\pi} \int_0^{\pi} f(x+t) D_k(t) dt,$$

where

$$D_k(t) = \frac{\sin\left(k + \frac{1}{2}\right)t}{2 \sin \frac{1}{2} t};$$

$D_k(t)$ is called the Dirichlet kernel.

Suppose now that the matrix $A = (a_{n,k})$ determines a regular summability method. We shall also assume that A is a triangular matrix, i.e., $a_{n,k} = 0$ for $k \geqq n+1$. The A transforms $t_n(x)$ of the partial sums $\{s_k(x)\}$ may then be written as follows:

$$t_n(x) = \sum_{k=1}^{n} a_{n,k} s_k(x) = \frac{2}{\pi} \sum_{k=1}^{n} a_{n,k} \int_0^{\pi} f(x+t) D_k(t) dt.$$

Regular matrices which satisfy the condition

$$\lim_{n \to \infty} \sum |a_{n,k} - a_{n,k+1}| = 0$$

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are called strongly regular [1; 2]. We shall employ a slightly stronger condition in the theorem that follows. The proof of this theorem parallels that of a similar theorem due to Hardy and Littlewood [3, p. 34].

THEOREM. *If*

(i) *A is a regular triangular matrix such that for some* $0<r<1$,

$$\lim_{n\to\infty} \sum k^r \left| a_{n,k} - a_{n,k+1} \right| = 0,$$

(ii) *$f(t)$ is Lebesgue integrable over* $(0, \pi)$ *and*

$$\left| f(x+h) - f(x) \right| = o\left[\left(\log \frac{1}{|h|}\right)^{-1}\right]$$

for some x then at the point x, the Fourier series of $f(t)$ is summable by A to $f(x)$.

PROOF. Clearly we may suppose that $f(t)$ is an even function and $f(x)=0$. We then have

$$\begin{aligned}
\left| \sum_{k=1}^{n} a_{n,k} \int_0^{\pi} f(x+t) D_k(t)dt \right| &\leqq \left| \sum_{k=1}^{n} a_{n,k} \int_0^{k^{-1}} f(x+t) D_k(t)dt \right| \\
&\quad + \left| \sum_{k=1}^{n} a_{n,k} \int_{k^{-1}}^{k^{-r}} f(x+t) D_k(t)dt \right| + \left| \sum_{k=1}^{n} a_{n,k} \int_{k^{-r}}^{\pi} f(x+t) D_k(t)dt \right| \\
&= \left| \sum_{k=1}^{n} a_{n,k} P_k(x) \right| + \left| \sum_{k=1}^{n} a_{n,k} Q_k(x) \right| + \left| \sum_{k=1}^{n} a_{n,k} R_k(x) \right|.
\end{aligned} \tag{3}$$

Since $|D_k(t)| \leqq Mk$ in $(0, k^{-1})$

$$|P_k(x)| = \left| \int_0^{k^{-1}} f(x+t) D_k(t)dt \right| \leqq Mk \int_0^{k^{-1}} |f(x+t)| dt \to 0$$

as $k\to\infty$, since $f(x+t)$ is continuous at x. Consequently $\{P_k(x)\}$ is a null sequence. Also $|D_k(t)| \leqq M'/t$ and

$$\begin{aligned}
|Q_k(x)| &= \left| \int_{k^{-1}}^{k^{-r}} f(x+t) D_k(t)dt \right| \leqq Mo[(\log |k|^r)^{-1}] \int_{k^{-1}}^{k^{-r}} \left| \frac{dt}{t} \right| \\
&= M'o[(\log |k|^r)^{-1}][\log |k| - r \log |k|].
\end{aligned}$$

Clearly $Q_k(x)$ is a null sequence and the first two sums in (3) can be made as small as we may wish by choosing n sufficiently large. For some δ, $|f(x+t)| <1$ for $0 \leqq t \leqq \delta$, and for large k such that $k^{-r} > \delta$

$$R_k(x) = \int_{k^{-r}}^{\delta} f(x+t) D_k(t)dt + \int_{\delta}^{\pi} f(x+t) D_k(t)dt = R_k'(x) + R_k''(x).$$

By the Riemann-Lebesgue theorem, $\{R_k''(x)\}$ is a null sequence and so is A summable to zero. We must now evaluate $|\sum a_{n,k} R_k'(x)|$. First we introduce the notation

$$D_0(t) + \cdots + D_k(t) = \frac{\sin^2 (k+1)(t/2)}{2 \sin^2 (t/2)} = K_k(t),$$

and observe that $|K_k(t)| \leqq M''/t^2$. For some k, $k=1, 2, \cdots, N$, $k^{-r} \geqq \delta$. However, since δ (and hence N) is fixed,

$$\lim_{n\to\infty} \sum_1^N |a_{n,k}| = 0.$$

Consider the matrix $(b_{n,k})$, $b_{n,k}=0$, $k=1, 2, \cdots, N$, $b_{n,k}=a_{n,k}$ elsewhere. The matrices $(b_{n,k})$ and $(a_{n,k})$ are equivalent, indeed for any sequence $\{s_k\}$

$$\lim_{n\to\infty} \sum (a_{n,k} - b_{n,k})s_k = 0.$$

Therefore in the sequel we shall assume without any loss of generality that $a_{n,k}=0$, $k=1, 2, \cdots, N$. We have for $n>N$

$$\begin{aligned} \left| \sum_{k=1}^n a_{n,k} R_k'(x) \right| &= \left| \sum_{k=N}^n a_{n,k} \int_{k^{-r}}^{\delta} f(x+t)[K_k(t) - K_{k-1}(t)]dt \right| \\ &\leqq \left| \sum_{k=N}^{n-1} (a_{n,k} - a_{n,k+1}) \int_{k^{-r}}^{\delta} f(x+t) K_k(t)dt \right| \\ &\quad + \left| \sum_{k=N+1}^{n} a_{n,k} \int_{k^{-r}}^{(k-1)^{-r}} f(x+t) K_k(t)dt \right|. \end{aligned}$$

It then follows that

$$\begin{aligned} &\left| \sum_{k=N}^{n-1} (a_{n,k} - a_{n,k+1}) \int_{k^{-r}}^{\delta} f(x+t) K_k(t)dt \right| \\ &\qquad \leqq \sum_{k=N}^{n-1} |a_{n,k} - a_{n,k+1}| \int_{k^{-r}}^{\delta} \left| \frac{M''}{t^2} \right| dt \\ &\qquad \leqq M'' \sum_{k=N}^{n-1} (k^r + \delta^{-1}) |a_{n,k} - a_{n,k+1}| \end{aligned}$$

and it is clear from the hypothesis

$$\lim_{n\to\infty} \sum_{k=N}^{n-1} (k^r + \delta^{-1}) \left| a_{n,k} - a_{n,k+1} \right| = 0.$$

Moreover,

$$\left| \sum_{k=N+1}^{n} a_{n,k} \int_{k^{-r}}^{(k-1)^{-r}} f(x+t)K_k(t)dt \right| \leqq \sum_{k=N+1}^{n} \left| a_{n,k} \right| \left| \int_{k^{-r}}^{(k-1)^{-r}} \frac{M''}{t^2} dt \right|$$

$$= M'' \sum_{k=N+1}^{n} \left| a_{n,k} \right| \left[(k-1)^r - k^r \right].$$

Now the matrix $(|a_{n,k}|)$ is not regular but does sum null sequences to zero if $(a_{n,k})$ is a regular matrix. The sequence $\{(k-1)^r - k^r\}$ is a null sequence for $0<r<1$ and so

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left| a_{n,k} \right| \left[k^r - (k+1)^r \right] = 0$$

and $\lim_{n\to\infty} \left| \sum_{k=1}^{n} a_{n,k} R_k'(x) \right| = 0$. This completes the proof of the theorem.

2. We have remarked before that there are functions of our class, even continuous functions, such that

$$\limsup_{k\to\infty} s_k(x) = \infty, \qquad \text{(see [3, p. 173])}.$$

Consider the matrix $B=(b_{n,k})$ where $b_{n,k}=1/n$ for $1\leqq k\leqq n-1$, $b_{n,n}=1/f(n)$, $b_{n,k}=0$ for $k\geqq n+1$, and $\lim_{n\to\infty} f(n)=\infty$. For any choice of $f(n)$ so that $\lim_{n\to\infty} b_{n,n}=0$, $B=(b_{n,k})$ is a strongly regular matrix. The Cesàro transform of a sequence is given by

$$t_n' = \frac{s_1 + \cdots + s_n}{n}.$$

The B transform of a sequence is given by

$$\tau_n = \sum_{k=1}^{n} b_{n,k} s_k = \frac{n-1}{n} t_{n-1}' + b_{n,n} s_n.$$

It is well known that the Cesàro method sums the Fourier series of any Lebesgue integrable function almost everywhere [3, p. 45]. We can choose $b_{n,n}$ so that $\lim_{n\to\infty} b_{n,n}=0$, but $\limsup_{n\to\infty} b_{n,n}s_n(x)=\infty$ for the partial sums of Fourier series of functions of the class of our theorem. For such a choice $\{\tau_n\}$ diverges, since $\{((n-1)/n)t_{n-1}'\}$ converges while $\{b_{n,n}s_n\}$ does not converge. In this way we see the class of matrices of our theorem cannot be extended to include all strongly regular matrices.

References

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