SOME CHARACTERIZATIONS OF RIEMANN n-SPHERES

ROBERT E. STONG

1. **Introduction.** The purpose of this paper is to extend a result of Feeman and Hsiung [1] characterizing the Riemann n-sphere. In addition, other similar characterizations are obtained.

Considerable use will be made of the following:

LEMMA. *Let V^{n+1} be a Riemannian manifold of dimension $n+1 \geqq 3$ and V^n a closed orientable hypersurface of class C^3 imbedded in V^{n+1}. Then at each point of V^n the mean curvatures $M_\alpha (M_0 = 1)$ satisfy*

$$M_\alpha^2 - M_{\alpha-1} M_{\alpha+1} \geqq 0 \tag{1.1}$$

for $\alpha = 1, \cdots, n-1$; if in addition $M_s, M_{s-1}, \cdots, M_{s-i}$ are positive

$$M_{s-1}/M_s \geqq M_{s-2}/M_{s-1} \geqq \cdots \geqq M_{s-i-1}/M_{s-i} \tag{1.2}$$

and if $M_s, M_{s-1}, \cdots, M_1$ are positive

$$M_1 \geqq M_2^{1/2} \geqq \cdots \geqq M_2^{1/s}. \tag{1.3}$$

If any of the above is an equality at every point of V^n, then every point is an umbilic and V^n is called a Riemann n-sphere.

2. V^{n+1} **of constant curvature.** Throughout this section, V^{n+1} will denote a Riemannian manifold of dimension $n+1 \geqq 3$ and constant Riemannian curvature K such that there is a normal coordinate system S of Riemann at a fixed point 0 covering the whole manifold. V^n denotes a closed orientable hypersurface of class C^3 imbedded in V^{n+1}.

Under these conditions, Feeman and Hsiung [1] have shown that

$$\int_{V^n} M_{\alpha-1} dA + \int_{V^n} M_\alpha p dA = 0 \tag{2.1}$$

for $\alpha = 1, \cdots, n$, where p is the scalar product of the unit normal vector e_{n+1} of the hypersurface V^n at the point P and the position vector Y of the point P with respect to the coordinate system S.

THEOREM 1. *If there is an integer s, $1 \leqq s \leqq n$ such that $M_s > 0$ and either $p \leqq -M_{s-1}/M_s$ or $p \geqq -M_{s-1}/M_s$ throughout V^n, then V^n is a Riemann n-sphere.*

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PROOF. The cases $1 \leqq s \leqq n-1$ were proven by Feeman and Hsiung [1]. Therefore only the case $s=n$ will be considered here. Applying formula (2.1) with $\alpha=n$,

$$\int_{V^n} (M_{n-1} + M_n p) dA = 0$$

and since $M_{n-1}+M_n p$ is of fixed sign, $p=-M_{n-1}/M_n$. Now applying (2.1) with $\alpha=n-1$,

$$\int_{V^n} M_{n-2} dA = -\int_{V^n} M_{n-1} p dA,$$

$$= \int_{V^n} M_{n-1}^2 / M_n dA,$$

or

$$\int_{V^n} (1/M_n)(M_{n-1}^2 - M_n M_{n-2}) dA = 0.$$

From (1.1), $(1/M_n)(M_{n-1}^2 - M_n M_{n-2}) \geqq 0$ so that $M_{n-1}^2 - M_n M_{n-2} = 0$ at all points of V^n. By the lemma, V^n is a Riemann n-sphere.

THEOREM 2. *If there are integers s and i, $1 \leqq i < s \leqq n$, with $M_s, \cdots, M_i > 0$ and constants $c_j \geqq 0$ for $i \leqq j \leqq s-1$ such that at all points of V^n one has $M_s = \sum c_j M_j$, then V^n is a Riemann n-sphere.*

PROOF. By equation (1.2),

$$(M_j/M_s - M_{j-1}/M_{s-1}) = (M_j/M_{s-1})(M_{s-1}/M_s - M_{j-1}/M_j) \geqq 0$$

for $i \leqq j \leqq s-1$, and equality holds everywhere only if V^n is a Riemann n-sphere. Thus

$$1 = \sum c_j (M_j/M_s) \geqq \sum c_j (M_{j-1}/M_{s-1})$$

or

$$M_{s-1} - \sum c_j M_{j-1} \geqq 0$$

with equality holding everywhere only if V^n is a Riemann n-sphere.

By (2.1),

$$\int_{V^n} (M_{s-1} - \sum c_j M_{j-1}) dA = -\int_{V^n} p(M_s - \sum c_j M_j) dA$$

$$= 0,$$

so that $M_{s-1} = \sum c_j M_{j-1}$ at all points of V^n.

COROLLARY. *If there are integers s and i, $1 \leqq i < s \leqq n$, with $M_s, \cdots, M_i > 0$ and a constant c with $M_s = cM_i$ at all points of V^n, then V^n is a Riemann n-sphere.*

THEOREM 3. *If there are integers s and i, $0 \leqq i < s < n$, with $M_{s+1}, \cdots, M_{i+1} > 0$ and constants $c_j \geqq 0$ for $i \leqq j \leqq s-1$ such that at all points of V^n, $M_s = \sum c_j M_j$ and if p is of fixed sign throughout V^n, then V^n is a Riemann n-sphere.*

PROOF. Applying the above procedure in reverse,

$$M_{s+1} - \sum c_j M_{j+1} \leqq 0$$

and vanishing identically only if V^n is a Riemann n-sphere. By equation (2.1)

$$\begin{aligned}\int_{V^n} (M_{s+1} - \sum c_j M_{j+1}) p\, dA &= -\int_{V^n} (M_s - \sum c_j M_j)\, dA \\ &= 0,\end{aligned}$$

and since $(M_{s+1} - \sum c_j M_{j+1})\ p$ is of fixed sign, $M_{s+1} = \sum c_j M_{j+1}$ everywhere.

THEOREM 4. *If there is an integer s, $1 < s \leqq n$, with $M_s > 0$ and a constant c with $M_s = cM_{s-1}$ at all points of V^n, then V^n is a Riemann n-sphere.*

PROOF. Since $M_s > 0$, c cannot be zero and M_{s-1} must be of fixed sign. By (1.1),

$$M_{s-1}(M_{s-1} - cM_{s-2}) = M_{s-1}^2 - M_s M_{s-2} \geqq 0,$$

so that $M_{s-1} - cM_{s-2}$ is of fixed sign and vanishes identically only if V^n is a Riemann n-sphere. Equation (2.1) gives

$$\begin{aligned}\int_{V^n} (M_{s-1} - cM_{s-2})\, dA &= \int_{V^n} (cM_{s-1} - M_s) p\, dA \\ &= 0,\end{aligned}$$

which implies $M_{s-1} = cM_{s-2}$ at all points of V^n.

COROLLARY. *If $M_n > 0$ and the sum of the principal radii of curvature is a constant, then V^n is a Riemann n-sphere.*

PROOF. If r_i denotes the ith principal radius of curvature, then

$$M_n = (n / \sum r_i) M_{n-1} = cM_{n-1}.$$

This corollary is a special case of a theorem of Chern [3]. A result similar to the following has been obtained by Chern, Hano, and Hsiung [4] for the symmetric functions of the principal radii of curvature.

THEOREM 5. *If there is an integer* s, $1<s\leqq n$, *with* $M_i>0$ *for* $i=1, \cdots, s$ *and a constant* c *with*

$$M_{s-1}^{1/(s-1)}\geqq c\geqq M_s^{1/s}$$

at all points of V^n, *and if* p *is of fixed sign throughout* V^n, *then* V^n *is a Riemann* n-*sphere.*

PROOF. Since p is of fixed sign, (2.1) with $\alpha=1$ implies that $p<0$. By (1.3), $M_1\geqq c$. Choose the orientation of V^n for which $g\geqq 0$ throughout V^n implies $\int_{V^n} g dA\geqq 0$. Using (2.1),

$$\begin{aligned} -\int_{V^n} c^{s-1}pM_1dA &\geqq -\int_{V^n} c^s p dA \\ &\geqq -\int_{V^n} M_s p dA \\ &= \int_{V^n} M_{s-1}dA \\ &\geqq \int_{V^n} c^{s-1}dA \\ &= -\int_{V^n} c^{s-1}pM_1dA. \end{aligned}$$

Hence all these terms are equal and

$$\int_{V^n} p(M_1-c)dA = 0$$

so $M_1=c$. By Theorem 3, with $s=1$ and $i=0$, V^n is a Riemann n-sphere.

THEOREM 6. *If there is an integer* s, $1<s\leqq n$, *with* M_s, $M_{s-1}>0$ *and a constant* c *with*

$$M_{s-1}/M_s\geqq c\geqq M_{s-2}/M_{s-1}$$

at every point of V^n, *and if* p *is of fixed sign throughout* V^n, *then* V^n *is a Riemann* n-*sphere.*

PROOF. From (2.1) with $\alpha=s$, one sees that $p<0$. Choose the orientation of V^n for which $g\geqq 0$ throughout V^n implies $\int_{V^n} g dA\geqq 0$. From (2.1),

$$
\begin{aligned}
\int_{V^n} M_{s-2}dA &= -\int_{V^n} pM_{s-1}dA \\
&\geqq -\int_{V^n} pcM_s dA \\
&= \int_{V^n} cM_{s-1}dA \\
&\geqq \int_{V^n} M_{s-2}dA.
\end{aligned}
$$

Hence, all these terms are equal and so

$$\int_{V^n} p(M_{s-1} - cM_s)dA = 0,$$

and $p(M_{s-1}-cM_s) \leqq 0$, which implies $M_{s-1}=cM_s$ at every point of V^n. By Theorem 4, V^n is therefore a Riemann n-sphere.

COROLLARY. *If S is a closed orientable surface of class C^3 twice differentiably imbedded in Euclidean 3-space and having $M_2>0$, then either* $\inf_{q\in S} M_1^2 < \sup_{q\in S} M_2$ *or S is a sphere.*

PROOF. By Hadamard's theorem, S is the boundary of a convex body, and choosing the origin in the interior of this body, p is of fixed sign. If $\inf M_1^2 \geqq \sup M_2$, there is a constant $c>0$ with $M_1^2 \geqq c^2 \geqq M_2$. Since M_1 is continuous, either $M_1 \geqq c$ or $M_1 \leqq -c$. In the first case, by Theorem 5, every point of S is an umbilic and so S is a sphere [2, p. 128]. In the second case, p must be positive and choosing the orientation as before,

$$
\begin{aligned}
-\int_S cpM_1 dA &= \int_S c dA \\
&\leqq -\int_S M_1 dA \\
&= \int_S pM_2 dA \\
&\leqq \int_S pc^2 dA \\
&\leqq -\int_S cpM_1 dA.
\end{aligned}
$$

Thus all these terms are equal and one has $M_2=-cM_1$. By Theorem 4, every point is an umbilic and S is a sphere.

3. $n+1\geqq 4$. In this section V^{n+1} will denote a Riemannian manifold of dimension $n+1\geqq 4$ and such that there is a normal coordinate system of Riemann at a fixed point 0 covering the whole manifold V^{n+1}. V^n is a closed orientable hypersurface of class C^3 imbedded in V^{n+1}.

With these assumptions, Feeman and Hsiung [1] have proven that

$$\int_{V^n} M_{\alpha-1}dA + \int_{V^n} M_\alpha p dA = 0 \tag{3.1}$$

for α odd and $1\leqq\alpha\leqq n$.

THEOREM 7. *If there is an even integer s, $1<s\leqq n$, with M_s, $M_{s-1}>0$ and $p\leqq -M_{s-1}/M_s$ at all points of V^n, then V^n is a Riemann n-sphere.*

PROOF. By (1.2), $p\leqq -M_{s-1}/M_s\leqq -M_{s-2}/M_{s-1}$ and by (3.1)

$$\int_{V^n}(M_{s-2}+pM_{s-1})dA = 0,$$

so $p\leqq -M_{s-1}/M_s\leqq -M_{s-2}/M_{s-1}=p$. Thus $M_{s-1}/M_s=M_{s-2}/M_{s-1}$ at all points of V^n, and by the lemma V^n is a Riemann n-sphere.

THEOREM 8. *If there is an even integer s, $1<s<n$, with M_s, $M_{s+1}>0$ and $p\geqq -M_{s-1}/M_s$ at all points of V^n, then V^n is a Riemann n-sphere.*

PROOF. By (1.2), $p\geqq -M_{s-1}/M_s\geqq -M_s/M_{s+1}$, and by (3.1),

$$\int_{V_n}(M_s+pM_{s+1})dA = 0,$$

so $p\geqq -M_{s-1}/M_s\geqq -M_s/M_{s+1}=p$. Therefore $M_{s-1}/M_s=M_s/M_{s+1}$ at all points of V^n, and V^n is a Riemann n-sphere.

For s odd, the corresponding results were obtained by Feeman and Hsiung [1].

Applying formula (3.1) to the integrations in Theorems 2 and 3 one obtains:

THEOREM 9. *If there are odd integers s and i, $1\leqq i<s\leqq n$, with $M_s, \cdots, M_i>0$ and constants $c_j\geqq 0$ ($c_j=0$ if j is even) for $i\leqq j\leqq s-2$ such that at all points of V^n one has $M_s=\sum c_jM_j$, then V^n is a Riemann n-sphere.*

THEOREM 10. *If there are even integers s and i, $0\leqq i<s<n$, with*

$M_{s+1}, \cdots, M_{i+1}>0$ and constants $c_j \geqq 0$ ($c_j=0$ if j is odd) for $i \leqq j \leqq s-2$ such that $M_s=\sum c_j M_j$ at every point of V^n, and if p is of fixed sign throughout V^n, then V^n is a Riemann n-sphere.

THEOREM 11. *If there is an odd integer s, $1<s\leqq n$, with $M_i>0$ for $i=1, \cdots, s$ and a constant c with $M_{s-1}^{1/(s-1)} \geqq c \geqq M_s^{1/s}$ at all points of V^n, and if p is of fixed sign throughout V^n, then V^n is a Riemann n-sphere.*

PROOF. The inequalities in the proof of Theorem 5 still hold, since by (3.1) the integrations can be performed. Thus

$$\int_{V^n} p(M_s - c^{s-1}M_1)dA = 0.$$

One has $c^{s-1}M_1 \geqq c^s \geqq M_s$ and p of fixed sign, so $M_s = c^{s-1}M_1$ everywhere. By Theorem 9, with $i=1$, $c_1=c^{s-1}$ and $c_j=0$ if $j\neq 1$, V^n is a Riemann n-sphere.

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UNIVERSITY OF OKLAHOMA