THE INFLUENCE OF THE DISSIPATIVE PART OF A
GENERAL MARKOV PROCESS

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0. Introduction. In this paper we give a complete solution of a problem posed by E. Hopf concerning the influence of the dissipative part of a general Markov process. Let $X$ be a space with points $x, y, \cdots$, and subsets $A, B, \cdots$, which form a field $\mathcal{F}$ and \{\(P^n(x, A), n \geq 1, x \in X, A \in \mathcal{F}\}\} and transition probabilities, of a general Markov process. They will be supposed to be countably additive in $A$, $\mathcal{F}$-measurable in $x$ and to satisfy the Chapman-Kolmogorov equation

\[ P^{n+1}(x, A) = \int_X P^1(x, A) P^n(x, dy). \]

The transition probabilities define related operators in various Banach spaces. We define for the Banach space of finite measures $m$ the transformation $T$ by the equation

\[ Tm(A) = \int_X P^1(x, A) m(dx). \]

We suppose that there exists a positive measure $m'$ on $\mathcal{F}$ such that the set $M_{m'}$ of $m'$-absolutely continuous measures is mapped into itself. It is known and trivial that this is essentially no restriction to impose on $T$. Let $L_1$ be the Banach space of all $m'$ integrable functions, and define a mapping $L^*$ of $L_1$ into itself by means of the following equation

\[ Tm''(A) = \int_A L^*f m'(dx), \]

if

\[ m''(A) = \int_A f m'(dx). \]

$L^*$ is clearly positive and has norm less than or equal to one.

It is known [2, Theorem 8.1] that $X$ splits into two disjoint sets $C$ and $D$, $X = C + D$, the conservative and the dissipative part, respectively, of $X$, such that if $\rho_1$ is in $L_1$, is non-negative, then

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\[
\sum_{0}^{\infty} L^{*k} p_1(x) < \infty, \quad \text{for almost all } x \text{ in } D,
\]
and such that if \( p_2 \) is in \( L_1 \), is positive, then
\[
\sum_{0}^{\infty} L^{*k} p_2(x) = \infty, \quad \text{for almost all } x \text{ in } C.
\]

We next consider the operations defined by
\[
L^* f = e_C L^* f, \quad L^* D f = e_D L^* f,
\]
where \( e_C \) and \( e_D \) are the characteristic functions of \( C \) and of \( D \), respectively. It can also be shown \([2, \text{Theorem 8.2}]\) that \( L^* D L^* = 0 \), but that in general the product of the operators in reverse order does not vanish. We introduce further notation:
\[
L_{D,n}^* = \sum_{0}^{n-1} L_{D}^{*k},
\]
\[
M_n^* = L_C L_{D,n}^*,
\]
\[
L_n^*(f, p) = \sum_{0}^{n-1} L^{*k} f \bigg/ \sum_{0}^{n-1} L^{*k} p.
\]

\( M_n^* \) measures the \( f \) contribution to \( C \) accumulated in \( n \) successive trials. It may easily be shown that \( M_n^* f = \lim_{n \to \infty} M_n^* f \) exists almost everywhere and is in \( L_1 \).

Hopf \([2, \text{p. 44}]\) asks the following question: is the limit of \( L_n^*(f-f', p) \) (as \( n \) tends to infinity) zero almost everywhere on \( C \), where \( f' = M_n^* f \)? We answer this question affirmatively. This means that in considering the limit of \( L_n^*(f, p) \) we may replace \( f \) by \( f' \) and \( p \) by \( p' \) and thus obtain a separation of the conservative and dissipative parts of \( X \).

1. Results and proofs. We state first the following lemmas:

**Lemma 1.** If \( L^* \) is a positive linear operator of \( L_1 \) into \( L_1 \) such that its norm is less than or equal to one, and if we define \( L^*(f, p) = \lim_{n \to \infty} L_n^*(f, p) \) for \( f \) and \( p \) in \( L_1 \) and \( p \) positive, then \( L^*(f, p) \) is well defined, and if \( \{f_n\} \) is a sequence of functions tending to zero in the \( L_1 \) norm, then the measure of the set where
\[
L^*(f_n, p) \geq a
\]
tends to zero as \( n \) tends to infinity, for each \( a > 0 \).
**Proof.** That $L^*(f, \rho)$ is well defined follows from [1, Theorem 1]. That the last property is satisfied follows from Hopf's maximal ergodic theorem, that
\[
\int_{A_n} f_n m'(dx) \geq a \int_{A_n} \rho m'(dx)
\]
where $A_n = \{x: \sup_{k \geq 1} L^*_k(f_n, \rho) \geq a\}$.

**Lemma 2.** If $L^*$ is as in Lemma 1.1 and if $f$ and $\rho$ are in $L_1$ and $\rho$ is positive, and if we define $g = f - L^*_i f$ for some fixed $i$, then
\[
\lim_{n \to \infty} L^*_n(g, \rho) = 0
\]
almost everywhere on $C$.

**Proof.** We may suppose without loss of generality that $f$ is non-negative. It follows from Lemma 1 of (1) that
\[
\lim_{n \to \infty} \frac{L^*_{n+1}f}{f + \cdots + L^*_n f} = 0
\]
almost everywhere on the set $B = C \cap \{x: \sum_0^\infty L^*_k f = \infty\}$. From this it follows that for each fixed $i$,
\[
\lim_{n \to \infty} L^*_n(L^*_i f, f) = 1
\]
almost everywhere on $B$. The proof of the lemma is complete on noting that on $C$, $\rho + \cdots + L^*_n \rho$ tends to infinity almost everywhere as $n$ tends to infinity, and thus that the lemma is trivial on that part of $C$ where $f + \cdots + L^*_n f$ does not tend to infinity.

**Theorem 1.** If $L^*$ satisfies the conditions of Lemma 1, and if $f' = M \omega f$ and if $\rho$ is positive and if $f$ and $\rho$ are in $L_1$ then
\[
\lim_{n \to \infty} L^*_n(f - f', \rho) = 0
\]
almost everywhere on $C$.

**Proof.** We note first that it follows directly from Lemma 2 that
\[
(1.1) \quad \lim_{n \to \infty} L^*_n(L^*_i f, \rho) - L^*_n(f, \rho) = 0 \text{ a.e. on } C.
\]
If we define $f_i = L^*_i \sum_{k=0}^{i-1} L^*_k f + L^*_i f$ we have, also from Lemma 2, that
\[
(1.2) \quad \lim_{n \to \infty} L^*_n(f_i, \rho) - L^*_n(L^*_i f, \rho) = 0 \text{ a.e. on } C,
\]
because \( f_i \) can be written as a sum of functions

\[
f_i = h_0 + h_1 + \cdots + h_i
\]
such that

\[
L^*f_i = L^*h_0 + L^*h_1 + \cdots + h_i,
\]
and this means that the difference of (1.2) is equal to (in terms of the \( h_i \))

\[
L_n^*(h_0, p) - L_n^*(L_n^*h_0, p) + L_n^*(h_1, p) - L_n^*(L_n^*h_1, p) + \cdots
\]

\[
+ L_n^*(h_i, p) - L_n^*(h_i, p)
\]

and each difference tends to zero separately (the last, of course, equals zero) by Lemma 2. To see that \( f_i \) can be written as such a sum, note that since

\[
f_i = L_c^* \sum_{k=0}^{i-1} L_D^k f + L_D^i f,
\]

\[
L^*f_i = (L_c^* + L_D^*)f = \sum_{k=0}^{i-1} L_c^* L_D^k f,
\]
and \((L_c^* L_D^* = 0)\)

\[
L^*f = (L_c^* + L_D^*)f = \sum_{k=0}^{i} L_c^* L_D^k f,
\]

we may take \( h_j = L_c^* L_D^j f, j = 0, 1, \cdots, i - 1, \) and \( h_i = L_D^i f. \) It follows easily from (1.1) and (1.2) that

\[
(1.3) \quad \lim_{n \to \infty} L_n^*(f_i, p) - L_n^*(f, p) = 0
\]
amost everywhere on \( C \) since the difference of (1.3) is equal to

\[
L_n^*(f_i, p) - L_n^*(L_n^*f_i, p) + L_n^*(L_n^*f_i, p) - L_n^*(f, p).
\]

We may, and do, suppose that \( f \) is non-negative, without loss of generality. We have \((L_D^* L_c^* = 0)\) that

\[
L^k L_D^i f = \sum_{j=0}^{k} L_c^j L_D^{i+k-j} f
\]
and thus that

\[
\sum_{k=0}^{n-1} L_c^k L_D^i f = \sum_{k=0}^{n-1} \sum_{j=0}^{k} L_c^j L_D^{i+k-j} f.
\]
Since
\[ \sum_{k=0}^{n-1} \sum_{j=0}^{k} L_C^j L_D^{i+k-j} f = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} L_C^j L_D^{i+k-j} f = \sum_{j=0}^{n-1} \sum_{k=0}^{n-(j+1)} L_C^j L_D^{i+k} f, \]
it follows that
\[ \sum_{k=0}^{n-1} L^*_L L_D^n f \leq \sum_{j=0}^{n} \sum_{k=0}^{n-(j+1)} L_C^j L_D^{i+k} f \]
and that on C,
\[ \sum_{k=0}^{n-1} L^*_L L_D^n f \leq \sum_{j=0}^{n} \sum_{k=0}^{n-(j+1)} L_C^j L_D^{i+k} f \leq \sum_{j=0}^{n} \sum_{k=0}^{\infty} L_C^j L_D^{i+k} f. \]
This implies that on C we have
\[
(1.4) \quad L_n^*(L_D^n f, p) \leq L_n^* \left( \sum_{k=0}^{\infty} L_C^k L_D^n f, p \right).
\]
Now, it follows from (1.4), since \( f' = \sum_{k=0}^{\infty} L_C^k L_D^n f, \) that
\[
| L_n^*(f', p) - L_n^*(f_i, p) | \leq \left| L_n^* \left( \sum_{k=0}^{\infty} L_C^k L_D^n f, p \right) - L_n^*(L_D^n f, p) \right| \leq 2 L_n^* \left( \sum_{k=0}^{\infty} L_C^k L_D^n f, p \right).
(1.5)
\]
on C, and thus that
\[
(1.6) \quad | L^*(f', p) - L^*(f_i, p) | \]
can be greater than a positive number \( a \) on a subset of C of measure which tends to zero as \( i \) tends to infinity, by Lemma 1. We now note that
\[
L^*(f' - f, p) = L^*(f', p) - L^*(f_i, p) + L^*(f_i, p) - L^*(f, p),
\]
and that the second difference is zero by (1.3) and that the first satisfies the condition listed above for (1.6). This clearly proves the theorem.

**Bibliography**


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