THE INFLUENCE OF THE DISSIPATIVE PART OF A GENERAL MARKOV PROCESS

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0. **Introduction.** In this paper we give a complete solution of a problem posed by E. Hopf concerning the influence of the dissipative part of a general Markov process. Let X be a space with points $x, y, \cdots$, and subsets $A, B, \cdots$, which form a field $\mathfrak{F}$ and $\{P^n(x, A), n \geqq 1, x \in X, A \in \mathfrak{F}\}$ and transition probabilities, of a general Markov process. They will be supposed to be countably additive in A, $\mathfrak{F}$-measurable in x and to satisfy the Chapman-Kolmogorov equation

$$P^{n+1}(x, A) = \int_X P^1(x, A)P^n(x, dy).$$

The transition probabilities define related operators in various Banach spaces. We define for the Banach space of finite measures m the transformation T by the equation

$$Tm(A) = \int_X P^1(x, A)m(dx).$$

We suppose that there exists a positive measure m' on $\mathfrak{F}$ such that the set $M_{m'}$ of m'-absolutely continuous measures is mapped into itself. It is known and trivial that this is essentially no restriction to impose on T. Let L_1 be the Banach space of all m' integrable functions, and define a mapping L^* of L_1 into itself by means of the following equation

$$Tm''(A) = \int_A L^*fm'(dx),$$

if

$$m''(A) = \int_A fm'(dx).$$

L^* is clearly positive and has norm less than or equal to one.

It is known [2, Theorem 8.1] that X splits into two disjoint sets C and D, $X = C + D$, the conservative and the dissipative part, respectively, of X, such that if p_1 is in L_1, is non-negative, then

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$$\sum_{0}^{\infty} L^{*k}p_1(x) < \infty, \qquad \text{for almost all } x \text{ in } D,$$

and such that if p_2 is in L_1, is positive, then

$$\sum_{0}^{\infty} L^{*k}p_2(x) = \infty, \qquad \text{for almost all } x \text{ in } C.$$

We next consider the operations defined by

$$L_C^* f = e_C L^* f,$$
$$L_D^* f = e_D L^* f,$$

where e_C and e_D are the characteristic functions of C and of D, respectively. It can also be shown [2, Theorem 8.2] that $L_D^* L_C^* = 0$, but that in general the product of the operators in reverse order does not vanish. We introduce further notation:

$$L_{D,n}^* = \sum_{0}^{n-1} L_D^{*k},$$

$$M_n^* = L_C^* L_{D,n}^*,$$

$$L_n^*(f, p) = \sum_{0}^{n-1} L^{*k} f \Big/ \sum_{0}^{n-1} L^{*k} p.$$

M_n^* measures the f contribution to C accumulated in n successive trials. It may easily be shown that $M_\infty^* f = \lim_{n\to\infty} M_n^* f$ exists almost everywhere and is in L_1.

Hopf [2, p. 44] asks the following question: is the limit of $L_n^*(f-f', p)$ (as n tends to infinity) zero almost everywhere on C, where $f' = M_\infty^* f$? We answer this question affirmatively. This means that in considering the limit of $L_n^*(f, p)$ we may replace f by f' and p by p' and thus obtain a separation of the conservative and dissipative parts of X.

1. **Results and proofs.** We state first the following lemmas:

LEMMA 1. *If L^* is a positive linear operator of L_1 into L_1 such that its norm is less than or equal to one, and if we define $L^*(f, p) = \lim_{n\to\infty} L_n^*(f, p)$ for f and p in L_1 and p positive, then $L^*(f, p)$ is well defined, and if $\{f_n\}$ is a sequence of functions tending to zero in the L_1 norm, then the measure of the set where*

$$L^*(f_n, p) \geqq a$$

tends to zero as n tends to infinity, for each $a > 0$.

PROOF. That $L^*(f, p)$ is well defined follows from [1, Theorem 1]. That the last property is satisfied follows from Hopf's maximal ergodic theorem, that

$$\int_{A_n} f_n m'(dx) \geqq a \int_{A_n} p m'(dx)$$

where $A_n = \{x: \sup_{k\geq 1} L_k^*(f_n, p) \geqq a\}$.

LEMMA 2. *If L^* is as in Lemma* 1.1 *and if f and p are in L_1 and p is positive, and if we define $g=f-L^{*i}f$ for some fixed i, then*

$$\lim_{n\to\infty} L_n^*(g, p) = 0$$

almost everywhere on C.

PROOF. We may suppose without loss of generality that f is non-negative. It follows from Lemma 1 of (1) that

$$\lim_{n\to\infty} \frac{L^{*n+i}f}{f + \cdots + L^{*n}f} = 0$$

almost everywhere on the set $B = C \cap \{x: \sum_0^\infty L^{*k}f = \infty\}$. From this it follows that for each fixed i,

$$\lim_{n\to\infty} L_n^*(L^{*i}f, f) = 1$$

almost everywhere on B. The proof of the lemma is complete on noting that on C, $p + \cdots + L^{*n}p$ tends to infinity almost everywhere as n tends to infinity, and thus that the lemma is trivial on that part of C where $f + \cdots + L^{*n}f$ does not tend to infinity.

THEOREM 1. *If L^* satisfies the conditions of Lemma* 1, *and if $f' = M_\infty^* f$ and if p is positive and if f and p are in L_1 then*

$$\lim_{n\to\infty} L_n^*(f - f', p) = 0$$

almost everywhere on C.

PROOF. We note first that it follows directly from Lemma 2 that

$$\lim_{n\to\infty} L_n^*(L^{*i}f, p) - L_n^*(\mathrm{f}, p) = 0 \text{ a.e. on } C. \tag{1.1}$$

If we define $f_i = L_C^* \sum_0^{i-1} L_D^{*k}f + L_D^{*i}f$ we have, also from Lemma 2, that

$$\lim_{n\to\infty} L_n^*(f_i, p) - L_n^*(L^{*i}f, p) = 0 \text{ a.e. on } C, \tag{1.2}$$

because f_i can be written as a sum of functions

$$f_i = h_0 + h_1 + \cdots + h_i$$

such that

$$L^{*i}f = L^{*i-1}h_0 + L^{*i-2}h_1 + \cdots + h_i,$$

and this means that the difference of (1.2) is equal to (in terms of the h_j)

$$\overset{*}{L}_n(h_0, p) - \overset{*}{L}_n(L^{*i-1}h_0, p) + \overset{*}{L}_n(h_1, p) - \overset{*}{L}_n(L^{*i-2}h_1, p) + \cdots + \overset{*}{L}_n(h_i, p) - \overset{*}{L}_n(h_i, p)$$

and each difference tends to zero separately (the last, of course, equals zero) by Lemma 2. To see that f_i can be written as such a sum, note that since

$$f_i = L_C^* \sum_0^{i-1} L_D^{*k} f + L_D^{*i} f,$$

$$L^{*i-(k+1)} L_C^* L_D^{*k} f = L_C^{*i-k} L_D^{*k} f,$$

and ($L_D^* L_C^* = 0$)

$$L^{*i} f = (L_C^* + L_D^*)^i f = \sum_{k=0}^{i} L_C^{*i-k} L_D^{*k} f,$$

we may take $h_j = L_C^* L_D^{*j} f$, $j = 0, 1, \cdots, i-1$, and $h_i = L_D^{*i} f$. It follows easily from (1.1) and (1.2) that

$$\lim_{n\to\infty} \overset{*}{L}_n(f_i, p) - \overset{*}{L}_n(f, p) = 0 \tag{1.3}$$

almost everywhere on C since the difference of (1.3) is equal to

$$\overset{*}{L}_n(f_i, p) - \overset{*}{L}_n(L^{*i} f, p) + \overset{*}{L}_n(L^{*i} f, p) - \overset{*}{L}_n(f, p).$$

We may, and do, suppose that f is non-negative, without loss of generality. We have ($L_D^* L_C^* = 0$) that

$$L^{*k} L_D^{*i} f = \sum_{j=0}^{k} L_C^{*j} L_D^{*i+k-j} f$$

and thus that

$$\sum_{k=0}^{n-1} L^{*k} L_D^{*i} f = \sum_{k=0}^{n-1} \sum_{j=0}^{k} L_C^{*j} L_D^{*i+k-j} f.$$

Since

$$\sum_{k=0}^{n-1}\sum_{j=0}^{k} L_C^{*j} L_D^{*i+k-j} f = \sum_{j=0}^{n-1}\sum_{k=j}^{n-1} L_C^{*j} L_D^{*i+k-j} f = \sum_{j=0}^{n-1}\sum_{k=0}^{n-(j+1)} L_C^{*j} L_D^{*i+k} f,$$

it follows that

$$\sum_{k=0}^{n-1} L^{*k} L_D^{*i} f \leqq \sum_{j=0}^{n}\sum_{k=0}^{n-(j+1)} L_C^{*j} L_D^{*i+k} f$$

and that on C,

$$\sum_{k=0}^{n-1} L^{*k} L_D^{*i} f \leqq \sum_{j=0}^{n-1} L_C^{*j} \sum_{k=0}^{n-(j+1)} L_C^{*} L_D^{*i+k} f \leqq \sum_{j=0}^{n-1} L^{*j} \sum_{k=0}^{\infty} L_C^{*} L_D^{*i+k} f.$$

This implies that on C we have

$$L_n^*(L_D^{*i} f, p) \leqq L_n^*\left(\sum_{k=i}^{\infty} L_C^* L_D^{*k} f, p\right). \tag{1.4}$$

Now, it follows from (1.4), since $f' = \sum_{k=0}^{\infty} L_C^* L_D^{*k} f$, that

$$\begin{aligned} |L_n^*(f', p) - L_n^*(f_i, p)| &\leqq \left| L_n^*\left(\sum_{k=i}^{\infty} L_C^* L_D^{*k} f, p\right) - L_n^*(L_D^{*i} f, p)\right| \\ &\leqq 2L_n^*\left(\sum_{k=i}^{\infty} L_C^* L_D^{*k} f, p\right) \end{aligned} \tag{1.5}$$

on C, and thus that

$$|L^*(f', p) - L^*(f_i, p)| \tag{1.6}$$

can be greater than a positive number a on a subset of C of measure which tends to zero as i tends to infinity, by Lemma 1. We now note that

$$L^*(f' - f, p) = L^*(f', p) - L^*(f_i, p) + L^*(f_i, p) - L^*(f, p),$$

and that the second difference is zero by (1.3) and that the first satisfies the condition listed above for (1.6). This clearly proves the theorem.

Bibliography

1. R. V. Chacon and D. S. Ornstein, *A general ergodic theorem*, Illinois J. Math. vol. 4 (1960) pp. 153–160.

2. E. Hopf, *The general temporally discrete Markoff process*, J. Math. Mech. vol. 3 (1954) pp. 13–45.

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