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ISOMETRIES OF GROUP ALGEBRAS[1]

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Let G be a locally compact abelian group, $\hat{G}$ its character group, and A the group algebra of G. Associated with any automorphism ϕ of A [2] is a homeomorphism τ of $\hat{G}$ onto itself, with the property that, for $\alpha \in \hat{G}$, $f \in A$ and $\mathfrak{F}$ the Fourier transform, $\mathfrak{F}(\phi(f))(\tau\alpha) = \mathfrak{F}(f)(\alpha)$. Results of Helson [2] and Wendel [4] state that if e is the unit of $\hat{G}$, then

$$(1) \qquad \tau(e)\tau(xy) = \tau(x)\tau(y), \qquad \text{for all } x, y \in \hat{G},$$

if and only if ϕ is an isometry. The object of the present note is to give a further equivalent form of the statement that ϕ is an isometry.

Let T_α, $\alpha \in \hat{G}$, be that operator on A which for all $x \in G, f \in A$ satisfies $(T_\alpha f)(x) = f(x)(x, \alpha)$. We consider homomorphisms ϕ of A onto A such that to each α there is a $\rho(\alpha) \in \hat{G}$ such that

$$(2) \qquad \phi T_\alpha = T_{\rho(\alpha)}\phi, \qquad \alpha \in \hat{G}.$$

Our result is that such homomorphisms are isomorphisms and indeed isometries.

THEOREM 1. *Let ϕ be a homomorphism of the group algebra A of the locally compact abelian group G onto itself. Suppose that ϕ satisfies* (2); *then ϕ is an isomorphism.*

Let K be the kernel of ϕ. Since ϕ is automatically continuous [3], K is a proper closed ideal of A. The Wiener Tauberian theorem thus yields a maximal regular ideal M containing K. From condition (2) it follows that for $k \in K$ and $\beta \in \hat{G}$, $\phi T_\beta(k) = T_{\rho(\beta)}\phi(k) = 0$, and there-

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fore $T_\beta(K)\subset K\subset M$. Let $k\in K$ and let α be the character corresponding to M; then $0=\mathfrak{F}(T_\beta(k))(\alpha)=\mathfrak{F}(k)(\beta\alpha)$ for arbitrary $\beta\in\hat{G}$. Consequently $k=0$, and ϕ is an isomorphism.

LEMMA. *Let ϕ be an automorphism of A which satisfies* (2), *and let τ be the associated homeomorphism of $\hat{G}$ onto itself. Then*:

(i) *The mapping ρ is a homomorphism of $\hat{G}$ onto itself.*

(ii) *For any α, $\beta\in\hat{G}$, $\tau(\alpha\beta)=\rho(\beta)\tau(\alpha)$.*

We show that ρ is a homomorphism into $\hat{G}$. That the image of $\hat{G}$ under ρ is all of $\hat{G}$ is then a consequence of (ii). For all $f\in A$, $T_{\rho(\alpha\beta)}\phi(f)=\phi T_{\alpha\beta}(f)=\phi T_\alpha T_\beta(f)=T_{\rho(\alpha)}\phi T_\beta(f)=T_{\rho(\alpha)}T_{\rho(\beta)}\phi(f)$. Since $\phi(f)$ ranges over A, $T_{\rho(\alpha\beta)}=T_{\rho(\alpha)}T_{\rho(\beta)}$, whence $\rho(\alpha\beta)=\rho(\alpha)\rho(\beta)$.

It follows from (2) that for all $\beta\in\hat{G}$, $\mathfrak{F}(\phi T_\alpha(f))(\tau(\beta))=\mathfrak{F}(T_{\rho(\alpha)}\phi(f))(\tau(\beta))$. Thus, $\mathfrak{F}(\phi(f))(\rho(\alpha)\tau(\beta))=\mathfrak{F}(T_{\rho(\alpha)}\phi(f)(\tau(\beta))=\mathfrak{F}(\phi T_\alpha(f))(\tau(\beta))=\mathfrak{F}(T_\alpha(f))(\beta)=\mathfrak{F}(f)(\alpha\beta)=\mathfrak{F}(\phi(f))(\tau(\alpha\beta))$. Since the Fourier transforms of elements in A separate the points of $\hat{G}$, we deduce that $\rho(\alpha)\tau(\beta)=\tau(\alpha\beta)$.

THEOREM 2. *Let A be the group algebra of a locally compact abelian group G, and let ϕ be an automorphism of A. The following conditions are equivalent*:

(a) *ϕ is an isometry.*

(b) *The associated homeomorphism τ of $\hat{G}$ onto itself satisfies* (1).

(c) *ϕ satisfies* (2).

In view of our earlier remarks it suffices to show that (b) is equivalent to (c). Suppose that ϕ satisfies (2). Let τ be the homeomorphism of $\hat{G}$ associated with ϕ. From (ii) of the lemma, with the identity e of $\hat{G}$ in the role of α, we obtain $\tau(\beta)=\rho(\beta)\tau(e)$ for all $\beta\in\hat{G}$. Thus, using the above and (i) of the lemma,

$$\tau(e)\tau(\alpha\beta)=(\tau(e))^2\rho(\alpha\beta)=\tau(e)\rho(\alpha)\tau(e)\rho(\beta)=\tau(\alpha)\tau(\beta).$$

Hence (c) implies (b).

Suppose that τ satisfies (1). Recall that

$$\mathfrak{F}(\phi(f))(\tau(\alpha))=\mathfrak{F}(f)(\alpha),\qquad \alpha\in\hat{G}, f\in A. \tag{3}$$

By repeated application of (3) and the definition of T_α, we obtain $\mathfrak{F}(\phi T_\alpha(f))(\tau(u))=\mathfrak{F}(T_\alpha(f))(u)=\mathfrak{F}(f)(\alpha u)=\mathfrak{F}(\phi(f))(\tau(\alpha u))$. If we now apply (1), the last term above becomes $\mathfrak{F}(\phi(f))(\tau(\alpha)\tau(u)(\tau(e))^{-1})=\mathfrak{F}(T_{\tau(\alpha)(\tau(e))^{-1}}\phi(f))(\tau(u))$. If we compare the initial and terminal points of the string of equalities, and note that the range of τ is $\hat{G}$, we see that

$$\mathfrak{F}(\phi T_\alpha(f))=\mathfrak{F}(T_{\tau(\alpha)(\tau(e))^{-1}}\phi(f)).$$

Thus $\phi T_\alpha(f) = T_{\tau(\alpha)(\tau(e))^{-1}}\phi(f)$. However f is arbitrary in A, so $\phi T_\alpha = T_{\tau(\alpha)(\tau(e))^{-1}}$ and thus we may take $\rho(\alpha) = \tau(\alpha)(\tau(e))^{-1}$. Hence ϕ satisfies (2), and (b) implies (c).

If the mapping ϕ is an involution on A, a result of Civin and Yood [1] yields a homeomorphism σ of period two of $\hat{G}$ onto itself such that $\mathfrak{F}(\phi(f))(\sigma\alpha) = [\mathfrak{F}(f)(\alpha)]^-$. Methods totally similar to the above and to those of [**2**; **4**] allow one to establish the following theorem.

THEOREM 3. *Let A be the group algebra of a locally compact abelian group, and let ϕ be an involution on A. The following conditions are equivalent.*

(a) *ϕ is an isometry.*

(b) *The associated homeomorphism σ of period two of $\hat{G}$ onto itself satisfies $\sigma(e)\sigma(xy) = \sigma(x)\sigma(y)$, for all $x, y \in \hat{G}$.*

(c) *ϕ satisfies* (2).

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