# AN EXAMPLE CONCERNING AFFINE CONNEXIONS

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**Introduction.** The purpose of this note is to exhibit a simply connected manifold on which is defined a complete affine connexion with the following property: for each integer $n>0$ there are two points such that any broken geodesic between them must contain at least $n$ breaks. The idea is to take a manifold, say $M$, with complete affine connexion such that the exponential map is not onto, remove neighborhoods of two points, getting $M'$, so that any geodesic (in $M$) leaving one of these neighborhoods must break at least once in order to intersect the other neighborhood, then attach a countable number of copies of $M'$ together through these neighborhoods. The difficulty arises in accomplishing the attaching while preserving completeness. Perhaps this process may prove useful in other studies. This example shows that Theorem 1 in [**3**] (see bibliography) does not hold if a bound is placed on the number of breaks a broken geodesic may have.

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1. **Local flattening of a connexion.** Let $M$ be a $C^\infty$ manifold on which is defined a complete affine connexion $\omega$, i.e., a connexion on the bundle of bases $B(M)$ (see [**1**]). Take a point $m$ in $M$ and let $\phi$ be a coordinate map with coordinate functions $x_1, \cdots, x_n$ on a neighborhood $U$ of $m$. We take $U$ so that the exponential map (for $\omega$) at $m$ is a diffeo with respect to $U$ and $U$ is convex (see [**4**]). Let $\Gamma$ denote the set of functions $\Gamma^i_{jk}$ belonging to $\omega$ and $\phi$; we assume $\phi$ is chosen so that all $\Gamma^i_{jk}(m)=0$. Further we suppose all $x_i(m)=0$ and $p$ in $U$ implies $\sum_1^n x_i(p)^2<R^2$. Let $d$ denote the distance function in $U$ induced by this coordinate system. For $0<r<R$, let $U(r)$ be the open ball about $m$ of radius $r$. In general, let $B(p, r) = [q\in U: d(p, q)<r]$.

THEOREM 1. *There is a complete connexion $\bar\omega$ on $M$ and a real number $s$ such that*: (1) $0<s<R$, (2) $\omega=\bar\omega$ *on* $M-U(s)$, (3) $\bar\omega$ *is the flat Riemannian connexion associated with $d$ in* $U(s/2)$.

PROOF. For each $r$, $0<r<R$, let $f_r$ be a real valued $C^\infty$ function on $M$ with $f_r(p)=0$ for $p$ in $U(r/2)$, $f_r(p)=1$ for $p$ in $M-U(r)$, and $0\leqq f_r\leqq 1$. Moreover we may assume the mapping: $(r, p)\to f_r(p)$ is continuous on the set $(0, R)\times M$. Let $f_0\equiv 1$ on $M$. For each $r$, $0\leqq r<R$,

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let $\omega_r$ be the connexion on $M$ defined by requiring $\omega_r = \omega$ on $M-U$, while $\Gamma_r = f_r\Gamma$ defines $\omega_r$ in terms of the coordinate map $\phi$ in $U$.

Note the function: $(r, p) \to f_{|r|}(p)\Gamma(p)$ is continuous on $(-R, R) \times U$. This is trivial at points where $r \neq 0$. For a point $(0, p)$ take any $\epsilon > 0$. If $p = m$, let $\delta > 0$ be chosen so that $q$ in $U(\delta)$ implies $|\Gamma^i_{jk}(q)| < \epsilon$ for all $i, j, k$. Then if $(r, q)$ in $(-R, R) \times U(\delta)$, $|f_{|r|}(q)\Gamma(q) - 0| < \epsilon$. If $p \neq m$, let $\delta_1 = d(m, p)/2$. Let $\delta_2 > 0$ be chosen so that $q$ in $B(p, \delta_2)$ implies $|\Gamma^i_{jk}(q) - \Gamma^i_{jk}(p)| < \epsilon$ for all $i, j, k$, and $\delta_2 < \delta_1$. Then if $(r, q)$ in $(-\delta_1, \delta_1) \times B(p, \delta_2)$, $|f_{|r|}(q)\Gamma(q) - \Gamma(p)| = |\Gamma(q) - \Gamma(p)| < \epsilon$ since $q$ in $M - U(\delta_1)$.

Let $T(M)$ be the tangent bundle to $M$; let $\pi: T(M) \to M$; let $W = [(p, X)$ in $T(M): p$ in $U(R/2)$ and $\exp_m(\pm X)$ in $U(R/2)]$; let $y_i = x_i \circ \pi$ and $y_{n+i} = dx_i$ for $i = 1, \cdots, n$. Then the differential equations for "lifted" geodesics for the $\omega_r$ connexion become (the prime denote differentiation with respect to $t$),

$$(y_i)' = y_{n+i}, \qquad (y_{n+i})' = -\sum_{jk} (f_r\Gamma^i_{jk} \circ \pi) y_{n+j} y_{n+k}$$

summing $j$ and $k$ from 1 to $n$. We abbreviate this system by writing $y' = F(y, r)$. Note $F$ is a continuous function on $D = W \times (-R/2, R/2)$ and satisfies a Lipschitz condition in $y$ uniformly on $D$.

Use the notation $g(t, (p, X), r)$ to denote a geodesic of the $\omega_r$ connexion defined on some interval about $t = 0$, with $0 \to p$ and tangent vector $X$ at $t = 0$. We know $g(t, (p, X), 0)$ is defined for all $t$ by completeness of $\omega$, but the image points may not all remain in $U$.

For each $(p, X)$ in $W$ we know $g(t, (p, X), 0)$ is defined and stays in $U$ for $t$ in $[-1, 1]$. A slight variation of a theorem from the theory of differential equations (see [2, p. 29]) gives us a $\delta_1 > 0$ such that if $|r| < \delta_1$ then $g(t, (p, X), r)$ is defined for $t$ in $[-1, 1]$. Now let

$$W' = \left[(p, X) \text{ in } W: \frac{3R}{8} \leqq d\left(m, \exp_p \frac{1}{2} X\right) \leqq \frac{R}{2}\right].$$

Let $V = U(R/8)$, let $W'' = W' \cap \pi^{-1}(\overline{V})$. Hence $W''$ is compact. The theorem from differential equations assures us $g(t, (p, X), r)$ is continuous on the set $(-1, 1) \times W \times (-\delta_1, \delta_1)$ and hence is uniformly continuous on $A = [-1/2, 1/2] \times W'' \times [-\delta_1/2, \delta_1/2]$. Thus for the number $R/16$ we obtain $\delta_2 > 0$ such that if we have elements $(t_1, (p, X_1), r_1)$ and $(t_2, (p_2, X_2), r_2)$, both in $A$, with $|t_1 - t_2| < \delta_2$, $d((p_1, X_1), (p_2, X_2)) < \delta_2$ and $|r_1 - r_2| < \delta_2$, then

$$d(g(t_1, (p_1, X_1), r_1), g(t_2, (p_2, X_2), r_2)) < \frac{R}{16} \cdot$$

Now take $s$ such that $0<s<\text{minimum}[\delta_1/2,\ \delta_2,\ R/8]$. Claim $\bar{\omega}=\omega_s$ is the desired connexion. We need only show $\bar{\omega}$ is complete.

Since $\bar{\omega}=\omega$ in $M-U(s)$ we need only worry about geodesics emanating from (or passing through) $U(s)$. Take $p$ in $U(s)$ and take any $Y$ tangent to $M$ at $p$. There is a real number $b$ such that $(p, bY)$ in $W''$, i.e., $3R/8\leqq d(m, \exp_p \frac{1}{2}bY)\leqq R/2$, for $U$ is convex with respect to the $\omega$ connexion. Then $g(t, (p, bY), s)$ is defined on $[-1, 1]$ and for $t=1/2$ we claim $d(m, g(\alpha))>s$ where $\alpha=(1/2, (p, bY), s)$, hence $g$ has reached a point at which $\bar{\omega}=\omega$ and hence may be indefinitely extended. The above inequality follows by letting $\beta=(1/2, (p, bY), 0)$, hence $\alpha$ and $\beta$ are both in $A$ and within $\delta_2$ of each other which implies $d(g(\alpha), g(\beta))<R/16$. But $d(m, g(\beta))\geqq 3R/8$, hence

$$d(m, g(\alpha)) \geqq \frac{3R}{8} - \frac{R}{16} = \frac{5}{16}R > \frac{R}{8} > s. \qquad \text{q.e.d.}$$

We remark if we take $\bar{\omega}=\omega_{s'}$ for any $0<s'<s$ the conclusion of Theorem 1 still holds.

2. **Attaching two spaces.** Let $M$ and $M'$ be two $C^\infty$ manifolds, with the same dimension $n$, each carrying complete connexions. We wish to define a process which can roughly be described as follows. We remove two particular neighborhoods $V$ and $V'$ from $M$ and $M'$ respectively. Let $P=(M-V)\cup N\cup(M'-V')$ where $N$ is a "neck" attached to $M-V$ along the boundary of $V$ and similarly attached to $M'-V'$. We then define a complete connexion on $P$ which coincides with the original connexions in $M-V$ and $M'-V'$.

Using the function $\exp(-(1/x)^2)$ we may construct a regular univalent $C^\infty$ function $\phi: E^1\to E^2$ such that for $t\geqq 1$, $\phi(t)=(t, 0)$; for $t\leqq -1$, $\phi(t)=(-t, 1)$; moreover, letting $\phi(t)=(x(t), y(t))$ we assume for $-1<t<1$ that $y(t)$ is strictly decreasing with $0<y(t)<1$; for $0<t<1$, $x(t)$ is strictly decreasing with $1/2<x(t)<1$, $x(0)=1/2$; and $x(-t)=x(t)$ for $-1<t<0$.

For any real number $r>0$ we define $N(r)$ to be the $n$-dimensional regular submanifold of $E^{n+1}$ consisting of the union of the three sets:

$$U_0 = \left[(a_1, \cdots, a_n, 0) : \sum_1^n a_i^2 \geqq r^2\right],$$

$$U_1 = \left[(a_1, \cdots, a_n, 1) : \sum_1^n a_i^2 \geqq r^2\right],$$

$$U_2 = \left[(a_1x(t), \cdots, a_nx(t), y(t)) : \sum_1^n a_i^2 = r^2 \text{ and } -1 \leqq t \leqq 1\right].$$

Using the Riemannian metric on $N(r)$ induced from the Euclidean metric on $E^{n+1}$, we obtain a complete connexion $\omega_0$ on $N(r)$ since $N(r)$ is complete (closed) as a subset of $E^{n+1}$.

Let $M$ and $M'$ be $C^\infty$ manifolds of dimension $n$ carrying complete affine connexions $\omega$ and $\omega'$, respectively. Let $m$, $m'$; $U$, $U'$; $x_i$, $x_i'$; $R$, $R'$; and $d$, $d'$ all be chosen as in §1 for $M$ and $M'$ respectively. Applying Theorem 1 to $M$ and $M'$ we obtain a connexion $\bar{\omega}$ on $M$ and number $s$, and a connexion $\bar{\omega}'$ on $M'$ and number $s'$. By taking the smaller we may assume $s=s'$ by the remark at the end of §1. Let $V=U(s/4)$, $V'=U'(s/4)$. Let $N(s/4)$ be the submanifold of $E^{n+1}$ defined above. Let $N=[p \text{ in } N(s/4): p \text{ in } U_2]$. Let $P=(M-V)\cup N \cup(M'-V')$. If $p$ is in the boundary of $V$ we identify $p$ with the point in $N$ with coordinates $(x_1(p), \cdots, x_n(p), 0)$; similarly, for $p'$ in the boundary of $V'$ we identify $p'$ with the point in $N$ with coordinates $(x_1'(p), \cdots, x_n'(p'), 1)$. Our definition of $N$ assures us $P$ is a $C^\infty$ manifold. Let $\varpi$ be the connexion defined on $P$ by $\varpi=\bar{\omega}$ on $M-V$, $\varpi=\omega_0$ on $N$, $\varpi=\bar{\omega}'$ on $M'-V'$. By Theorem 1 these connexions agree on the "overlap." Furthermore $\varpi$ is complete and $\varpi=\omega$ on $M-U(s)$, $\varpi=\omega'$ on $M'-U'(s)$.

We refer to the above process by saying we have attached $M$ and $M'$ through the neighborhoods $U$ and $U'$ respectively.

3. **The example.** Let $G$ be the Lie group $SL(2, C)=[2\times 2$ matrices $T$ over $C$: $\det T=1]$. Let $\omega$ be the complete connexion on $G$ defined by left translation. We provide two points $p_1$ and $p_2$ in $G$, with neighborhoods $U_1$ and $U_2$ respectively, such that any geodesic leaving $U_1$ must break at least once in order to intersect $U_2$. Then going to the simply connected covering and "inverting" through these neighborhoods we obtain the example.

We recall that the exponential map from the Lie algebra of $G$ into $G$ does not map onto $G$, for any element in the image has a square root and any element in this group having a square root has trace $\geqq -2$, however there exist elements with trace $<-2$. Let $p$ be in $G$ with trace $p<-2$, let $e$ be the identity. Since trace is continuous we may let $U$ be a neighborhood of $p$ such that trace $(U)<-2$. We further take $U=V_1p$ where $V_1$ is a neighborhood of $e$. Let $V$ be a neighborhood of $e$ with $VV\subset V_1$ and $V=V^{-1}$. We claim no unbroken geodesic emanating from a point in $V$ intersects $Vp$. For if $g$ was such a geodesic, say $g(0)=m$ in $V$, $g(a)=p'$ in $Vp$, then $m^{-1}g(0)=e$, $m^{-1}g(a)=m^{-1}p'=m^{-1}m'p$ is in $U$. But $m^{-1}g$ is a geodesic and no unbroken geodesic passing through $e$ can intersect $U$ because of the trace condition. Let $U_1=V$, $U_2=Vp$.

For each integer $n\geqq 1$, let $M_n=G$. By §2 we may attach each $M_k$

to $M_{k+1}$, for $k \geqq 1$, through the neighborhoods $U_2$ and $U_1$ respectively, obtaining a manifold $N$ with complete connexion which agrees with $\omega$ on each $M_k-(U_1 \cup U_2)$, $k>1$. Hence for any integer $n$ we need only let $p_0=e$ in $M_1-U_2 \subset N$ and let $p_n$ be any point in $M_n-(U_1 \cup U_2)$, then any broken geodesic from $p_0$ to $p_n$ must break at least $n$ times. Finally, let $M$ be the simply connected covering of $N$, let $\pi: M \rightarrow N$ be the covering map, and define a connexion $\omega^*$ on $M$ by $\omega^*=\pi^*\omega$. For each $i \geqq 0$, let $m_i$ be a point in $M$ with $\pi(m_i)=p_i$. Then any broken geodesic from $m_0$ to $m_n$ must contain at least $n$ breaks since $\pi$ carries a geodesic in $M$ into a geodesic in $G$.

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