A REPRESENTATION THEOREM FOR BOUNDED CONVEX SETS

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1. **Introduction.** It is well known that any closed convex subset of a normed linear space can be represented as the intersection of all those closed half-spaces which contain it. It is also well known (and easy to prove) that if the closed convex set is *bounded*, and if the space under consideration is Euclidean n-space, then the set can be represented as the *intersection of cells*, i.e., the following is true in the case when E is Euclidean n-space:

If C is a bounded closed convex subset of the normed linear space E, and if $x \in E \sim C$, then there exist $y \in E$ and $r > 0$ such that the cell $N_r y = \{z: \|z-y\| \leqq r\}$ contains C but not x.

In the present paper we consider those spaces E for which the above statement holds for all such C; such a space is said to have the *property* $(\mathcal{I})$: *Every bounded closed convex set C can be represented as the intersection of cells.*

Our study was motivated in part by a paper of S. Mazur [5], in which property $(\mathcal{I})$ was shown to hold for all reflexive Banach spaces having a strongly differentiable norm, and in which the following criterion for weak sequential convergence was proved:

Suppose E has property $(\mathcal{I})$. Then a sequence $\{x_n\}$ in E converges weakly to the point x if and only if (i) *the sequence $\{x_n\}$ is bounded and* (ii) *every cell containing infinitely many x_n also contains x.*

In what follows we obtain a general sufficient condition for property $(\mathcal{I})$, as well as some necessary conditions, these conditions all being given in the form of density criteria for certain subsets of U^*, the unit cell of the conjugate space E^*. For finite dimensional E, these conditions are equivalent, and we obtain the following interesting result:

A finite dimensional space E has property $(\mathcal{I})$ if and only if the set of extreme points of U^* is dense in the boundary of U^*.

An example is given which shows that in infinite dimensional spaces, the above condition (while necessary) does not imply property $(\mathcal{I})$. The same example provides an answer to a question of Mazur [5], by showing the existence of a Banach space in which the norm is

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weakly differentiable *everywhere* (except at the origin) but *strongly* differentiable *nowhere.*

Before proceeding, we list some definitions used throughout the paper. Denote the set of real numbers by R. Let S and S^* denote the unit spheres of E and E^* respectively, i.e., those elements of norm 1, and let U and U^* denote the unit *cells* of E and E^*, i.e., those elements of norm at most 1.

2. **A sufficient condition for property** $(\mathcal{I})$. If C is a bounded, closed, convex subset of a normed linear space E such that $\phi \notin C$, let C' be the set of all linear functionals f in E^* such that

$$\inf f(C)(=\inf\{f(x): x \in C\})$$

is positive. It follows from the basic separation theorem [1] that C' is nonempty. A subset K of a linear space is called a *convex cone* if K is convex and if $x \in K$ and $\lambda > 0$ imply $\lambda x \in K$.

LEMMA 2.1. *If C is a bounded, closed convex subset of a normed linear space E, then C' is an open convex cone in E^*.*

PROOF. It is easily verified that C' is a convex cone. To see that C' is open, suppose that $f \in C'$, let $\alpha = \inf f(C) > 0$, and choose $M > 0$ such that $C \subset N_M\phi$. If $\|f-g\| < \alpha/2M$, then for each $x \in C$ we have $f(x) - g(x) \leqq \|f-g\| \, \|x\| < \alpha/2$, so that $g(x) \geqq f(x) - \alpha/2 \geqq \alpha/2 > 0$. Hence $\inf g(C) > 0$, which shows that $g \in C'$ and therefore C' is open.

We say that the norm in E is *strongly differentiable* at the point $x \neq \phi$ if there exists an f in S^* such that $\lim_{y \to 0} \delta_x(y)/\|y\| = 0$, where $\delta_x(y) = \|x+y\| - \|x\| - f(y)$. Let str S^* be the set of all f in S^* which attain their norm at a point of strong differentiability, i.e., for which there exists an x in S such that the norm in E is strongly differentiable at x and $f(x) = 1$. The following theorem, while more general than Mazur's theorem [5], is proved in much the same way.

THEOREM 2.2. *A normed linear space E has property $(\mathcal{I})$ if* str S^* *is dense in S^*.*

PROOF. Suppose that C is a bounded, closed convex subset of E and that $y \in E \sim C$. We may assume, without loss of generality, that $y = \phi$. Since str S^* is dense in S^* and (by Lemma 2.1) $C' \cap S^*$ is open in S^*, there exists an f in str S^* and an x in S such that $\inf f(C) > 0$, $f(x) = 1$, and the norm in E is strongly differentiable at x. Choose $\epsilon > 0$ such that $\inf f(C) > 2\epsilon > 0$ and let $z = \epsilon x$. For each $r > 1$ let N_r be the cell of radius $\|(r-1)z\| = (r-1)\epsilon$ which is centered at rz. The origin is not in N_r for any $r > 1$, so it will suffice to show that $C \subset N_r$ for some $r > 1$. Suppose not; then there exist sequences $\{r_n\}$ and $\{x_n\}$

such that $r_n > 1$, $r_n \to \infty$, $\|x_n - r_n z\| > (r_n - 1)\epsilon$ and $x_n \in C$ for each n. The sequence $\{x_n\}$ is bounded because C is bounded, so we have $y_n = -x_n/\epsilon r_n \to \phi$ as $n \to \infty$. Hence, by strong differentiability of the norm at x, we must have $\delta_x(y_n)/\|y_n\| \to 0$ as $n \to \infty$. But $\epsilon r_n \delta_x(y_n) = \|r_n z - x_n\| - \epsilon r_n + f(x_n) > (r_n - 1)\epsilon - \epsilon r_n + 2\epsilon = \epsilon$. Hence

$$\delta_x(y_n)/\|y_n\| = \epsilon r_n \delta_x(y_n)/\|x_n\| > \epsilon/\|x_n\|,$$

a contradiction, since the x_n are bounded.

If E is a normed linear space let $P(E)$ be the set of all those linear functionals f in E^* which attain their supremum on S, i.e. those f for which there exists x in S such that $f(x) = \|f\|$. We call E *subreflexive* if $P(E)$ is norm-dense in E^*, or equivalently, if $P(E) \cap S^*$ is dense in S^*. Note that, always, str $S^* \subset P(E) \cap S^*$, so it follows immediately from our hypothesis in Theorem 2.2 that E is subreflexive. In Theorem 4.3 (i) it will be shown that subreflexivity is a necessary condition of property ($\mathcal{I}$). (There exist non-subreflexive normed linear spaces [**6**; see also **7**] but, to the best of our knowledge, the question "Is every *Banach* space subreflexive?" remains open.)[1]

3. **An elementary lemma.** In this section we prove a lemma which describes a basic connection between pairs of linear functionals and the hyperplanes determined by them. Roughly speaking, the lemma asserts the intuitively obvious fact that if two functionals f and g of norm one have their hyperplanes $f^{-1}(0)$ and $g^{-1}(0)$ sufficiently close together, then one of $\|f-g\|$, $\|f+g\|$ must be small.

LEMMA 3.1. *Suppose that E is a normed linear space and that $\epsilon > 0$. If $f, g \in S^*$ are such that $f^{-1}(0) \cap U \subset g^{-1}[-\epsilon/2, \epsilon/2]$, then either $\|f-g\| \leqq \epsilon$ or $\|f+g\| \leqq \epsilon$.*[2]

PROOF. By the Hahn-Banach theorem we can choose h in E^* such that $h = g$ on $f^{-1}(0)$ and $\|h\| = \sup |g(U \cap f^{-1}(0))|$. Then, by hypothesis, $\|h\| \leqq \epsilon/2$. Furthermore, since $g - h$ vanishes on $f^{-1}(0)$ there exists α in R such that $g - h = \alpha f$. Hence $\|g - \alpha f\| = \|h\| \leqq \epsilon/2$. Assuming $\alpha \geqq 0$, we will show that $\|f - g\| \leqq \epsilon$. (Otherwise, the same proof applied to $(-\alpha)f$ would show that $\|f+g\| \leqq \epsilon$.) If $\alpha \geqq 1$, then $\alpha^{-1} \leqq 1$ and $\|g - f\| = \|(1-\alpha^{-1})g + \alpha^{-1}(g - \alpha f)\| \leqq 1 - \alpha^{-1} + \alpha^{-1}\|g - \alpha f\|$. Also, $\alpha = \|\alpha f\| \leqq \|g\| + \|g - \alpha f\|$ so $1 - \alpha^{-1} \leqq (1 + \|g - \alpha f\|)^{-1}\|g - \alpha f\| \leqq \|g - \alpha f\|$. Hence $\|g - f\| \leqq 2\|g - \alpha f\| \leqq \epsilon$. If $0 \leqq \alpha < 1$, then $\|g - f\| \leqq \|g - \alpha f\| + \|(1-\alpha)f\| = \|g - \alpha f\| + 1 - \alpha = \|g - \alpha f\| + \|g\| - \|\alpha f\| \leqq 2\|g - \alpha f\| \leqq \epsilon$, which completes the proof.

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[1] *Note added in proof*: See Errett Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc., to appear.

[2] It should be noted that this lemma provides a short proof of Theorem 1.2 of [**6**].

4. Some consequences of property ($\mathcal{J}$). If x is in S, let F_x be the *face of* S^* *determined by* x, i.e., $F_x = \{f: \|f\| = 1 = f(x)\}$. Each such face is closed, convex and (by the Hahn-Banach theorem) nonempty. The next lemma is basic to the rest of the paper.

LEMMA 4.1. *Suppose that the normed linear space E has property* ($\mathcal{J}$). *If $f \in S^*$ and $0 < \epsilon < 1$ there exist $\delta(\epsilon) > 0$ and $x \in S$ such that $y \in S \cap N_\delta x$ implies $F_y \subset N_\epsilon f$.*

PROOF. If $f \in S^*$ and $0 < \epsilon < 1$ let $D = U \cap f^{-1}(0)$ and pick $u \in S$ such that $f(u) > \epsilon$. Let $u' = (\epsilon/2)u$. Since D is bounded, closed and convex, and $u' \notin D$, property ($\mathcal{J}$) implies the existence of $r > 0$ and $z \in E$ such that $u' \notin N_r z \supset D$. Let w be the intersection of the segment $[u', z]$ with the boundary of $N_r z$; then if C denotes the convex hull of u' and $N_r z$ we can conclude that $w \in \text{int } C$, the interior of C. Hence there exists $\alpha > 0$ such that $\|v - w\| \leqq \alpha$ implies $v \in \text{int } C$. Note that for any such v there exists $v' \in \text{int } N_r z$ and $\lambda \in]0, 1[$ such that $v = \lambda u' + (1 - \lambda)v'$.

Now let $x = r^{-1}(w - z)$ and let $\delta = \alpha r^{-1}$. If $y \in S \cap N_\delta x$ and $g \in F_y$ then $g(y) = \|g\| = 1 = \sup g(U)$. Letting $v = ry + z$ we have $g(v) = r \sup g(U) + g(z) = \sup g(N_r z)$. Now $\|v - w\| = \|v - (rx + z)\| = r\|x - y\| \leqq r\delta = \alpha$, so there exist v' and λ as above. Since $v' \in \text{int } N_r z$, we have $g(v') < g(v)$, while $\phi \in D \subset N_r z$ implies $g(v) \geqq 0$. Hence $g(v) = \lambda g(u') + (1 - \lambda)g(v') < \lambda g(u') + (1 - \lambda)g(v)$ or $0 \leqq g(v) < g(u')$. Since $g(u) \leqq \|g\| \cdot \|u\| = 1$, $g(u') = (\epsilon/2)g(u) \leqq \epsilon/2$ and therefore $g(v) < \epsilon/2$. Thus, for any $w' \in D \subset N_r z$, $g(w') \leqq g(v) < \epsilon/2$ and, by the symmetry of D about ϕ, $g(w') > -\epsilon/2$. Hence $D = f^{-1}(0) \cap U \subset g^{-1}[-\epsilon/2, \epsilon/2]$ and therefore, by Lemma 3.1, either $\|f - g\| \leqq \epsilon$ or $\|f + g\| \leqq \epsilon$. But $g(u) > (2/\epsilon)g(v) \geqq 0$ and $f(u) > \epsilon$, so $\epsilon < (f + g)(u) \leqq \|f + g\|$, which shows that $\|f - g\| \leqq \epsilon$. Since this is true for any $g \in F_y$, we have $F_y \subset N_\epsilon f$.

If K is a convex subset of a linear space we say that a subset A of K is an *extremal* subset of K if A is nonempty and both A and $K \sim A$ are convex. An extremal subset of K which consists of a single point is called an *extreme point* of K. It is easy to verify that if $x \in S$ then F_x is an extremal subset of U^*, the unit cell of E^*. Furthermore, the set $\{f: f(x) = 1\}$ is weak*-closed in E^* and hence, since U^* is weak*-compact, $F_x = U^* \cap \{f: f(x) = 1\}$ is weak*-compact. It follows from the Krein-Milman theorem [1] that F_x has at least one extreme point, and (using the fact that F_x is extremal in U^*) this point is also an extreme point of U^*. If B is a convex set, let ext B be the set of all extreme points of B.

Let φ be a "selection" for the mapping $x \to F_x$ such that, for each $x \in S$, $\varphi(x)$ is an extreme point of F_x. If A is any subset of S, then the cardinality of $\varphi(A) = \{\varphi(x): x \in A\}$ is no greater than that of A. We have the following immediate (and useful) corollary to Lemma 4.1.

COROLLARY 4.2. *If E is a normed space having property* ($\mathcal{I}$), *and if A is dense in S, then $\varphi(A)$ is dense in S^*.*

We say that $x \in S$ is a *smooth point* of S if F_x consists of exactly one point. Denote the set of smooth points of S by sm S. A point y of a convex set $K \subset E$ is called an *exposed point* of K if there exists a hyperplane H supporting K at y such that $H \cap K = \{y\}$. Equivalently, $y \in \exp K$ (the set of all exposed points of K) if and only if there exists a linear functional $f \in E^*$ such that $\sup f(K) = f(y)$ and $f(z) < f(y)$ for $z \in K \sim \{y\}$. Note that, always, $\exp K \subset \operatorname{ext} K$. If $x \in \text{sm } S$ and f is the unique point of F_x, consider x as a continuous linear functional on E^* (by letting $x(g) = g(x)$ for each $g \in E^*$); then if $g \in U^* \sim \{f\}$, $x(g) < 1 = x(f)$, so $f \in \exp U^*$. Thus, the following inclusions always hold: $\varphi(\text{sm } S) \subset \exp U^* \subset \operatorname{ext} U^*$.

THEOREM 4.3. *Suppose that E is a normed linear space having property* ($\mathcal{I}$); *then the following implications are valid*:

(i) *The set $\varphi(S)$ ($\subset P(E) \cap$ ext U^*) is dense in S^*; hence E is subreflexive and* ext U^* *is dense in S^*.*

(ii) *If* sm S *is dense in S, then* $\varphi(\text{sm } S)$ ($\subset \exp U^*$) *is dense in S^*.*

(iii) *If E is separable, then E^* is separable.*

(iv) *If E is separable and complete, then* exp U^* *is dense in S^*.*

PROOF. All the statements, with the exception of (iv), follow trivially from Corollary 4.2 and the preceding remarks and definitions. To prove (iv), we use the theorem of Mazur [**4**] which states that, for separable Banach spaces, sm S is dense in S; (ii) now yields (iv).

THEOREM 4.4. *If E is a finite dimensional normed linear space the following assertions are equivalent*:

(i) *The space E has property* ($\mathcal{I}$).

(ii) *The set* exp U^* *is dense in S^*.*

(iii) *The set* ext U^* *is dense in S^*.*

PROOF. That (i) implies (iii) follows from Theorem 4.3 (i). By a theorem of Klee [**3**, Theorem 2.3], exp U^* is dense in ext U^*, so (iii) implies (ii). To show that (ii) implies (i), we first show that $\exp U^* = \text{str } S^*$ in any finite dimensional space, then we apply Theorem 2.2. Mazur has pointed out [**5**, p. 130] that, in a finite dimensional space E, a point x is in sm S if and only if the norm in E is strongly differentiable at x. Thus, if $f \in \text{str } S^*$, f attains its supremum at a smooth point of S and therefore, using the remarks preceding Theorem 4.3, $f \in \exp U^*$. On the other hand, if $f \in \exp U^*$, there exists X in E^{**} such that $X(f) = 1 = \|X\|$ and $X(g) < 1$ for $g \in U^*$

$\sim\{f\}$. From the reflexivity of finite dimensional spaces it follows that there exists an $x \in S$ such that $g(x) = X(g)$ for each $g \in E^*$. From the above conditions on X we conclude that F_x is the single point f, i.e., $x \in \text{sm } S$. Thus, the norm in E is strongly differentiable at x and therefore $f \in \text{str } S^*$, which was to be shown.

If E is two-dimensional it is easy to verify that ext U^* is dense in S^* if and only if ext $U^* = S^*$, i.e., if and only if E^* is *strictly convex*. This property is equivalent (see e.g. [6]) to the property that E itself is *smooth*, i.e., that sm $S = S$. Thus we have the following corollary to Theorem 3.5.

COROLLARY 3.6. *A two-dimensional normed linear space E has property ($\mathcal{G}$) if and only if E is smooth.*

5. **An example.** In this section we exhibit a separable Banach space E which is isomorphic to l_1 (the space of absolutely summable sequences) and which has the following properties:

(i) E is smooth and E^* is strictly convex.

(ii) E is subreflexive.[1]

(iii) The set str S^* is empty, hence the norm in E is not strongly differentiable at any point of S.

(iv) The set exp U^* is dense in S^*.

(v) E does *not* have property ($\mathcal{G}$).

Now, smoothness of E is equivalent to *weak differentiability* of the norm at each $x \neq \phi$ in E (see, e.g. [1]), hence properties (i) and (iii) give a negative answer to the question, noted by Mazur in [5], as to whether weak differentiability always implies strong differentiability of the norm.

We will obtain the space E by using a theorem of Day's [2, Theorem 5] to renorm l_1. We first define a new norm for $m = l_1^*$ (the space of bounded sequences $y = \{y_i\}$) by $\|y\| = \sup |y_i| + (\sum y_i^2/2^i)^{1/2}$. If $x = \{x_i\}$ is in l_1, let $\|x\| = \sup \{\sum x_i y_i : y = \{y_i\} \in m \text{ and } \|y_i\| = 1\}$. [To see how Day's theorem is applied to yield these norms, let the space B_0 of his theorem be l_2, let B be l_1 and let T be the mapping from l_2 into l_1 defined by $T(\{x_i\}) = \{x_i/2^{i/2}\}$.] Then, letting E be l_1 so renormed, we can conclude from Day's theorem that E is isomorphic with l_1, that E^* is the space of bounded sequences with the norm described above, that E is smooth and that E^* is strictly convex. Thus, E has property (i) above.

To prove that str S^* is empty, we first prove the following simple lemma:

If the norm in a linear space E is strongly differentiable at the point $x \neq \phi$, if $f \in S^*$ is such that $f(x) = \|x\|$, if $\{f_n\} \subset E^*$ is such that

$f_n(x)\to f(x)$ and $\|f_n\|\to\|f\|=1$, then $\|f_n-f\|\to 0$. To see this, note first that we can assume $\|f_n\|=1$ for all but a finite number of n, since the lemma is true for the sequence $\{f_n\}$ if and only if it is true for the sequence $\{f_n/\|f_n\|: \|f_n\|\neq 0\}$. Now if $\|f_n-f\|\not\to 0$ we can choose (taking a subsequence if necessary) $\epsilon>0$ and $y_n\in S$ such that $(f_n-f)(y_n)\geqq 2\epsilon$. Now, let $x_n=\epsilon^{-1}[\|x\|-f_n(x)]y_n$ and use the fact that $\|x+x_n\|\geqq f_n(x)+f_n(x_n)$ to show that $\|x_n\|^{-1}[\|x+x_n\|-\|x\|-f(x_n)]\geqq\epsilon$, contradicting the assumption of strong differentiability at x and the fact that $\|x_n\|\to 0$.

Thus, in order to show that no point $x\neq\phi$ in E is a point of strong differentiability, it suffices to show that for each $x\neq\phi$ in E and any $y\in S^*$ such that $(x, y)=\|x\|$ there exists a sequence $\{y^k\}\subset E^*$ such that $(x, y^k)\to\|x\|$, $\|y^k\|\to\|y\|$, but $\|y^k-y\|\not\to 0$. (The latter is equivalent, of course, to $\lim_{k\to\infty}\sup_i|y_i^k-y_i|\neq 0$.) Assuming we have such x and y, we consider two cases. First, suppose that $y_i\to 0$. Pick an integer n such that $|y_i|<1/4$ if $i>n$ and define $y^k\in E^*(k>n)$ by $y_i^k=y_i$ if $i\neq k$, $y_k^k=3/8$. Then $(x, y^k)=(x, y)-x_k(y_k-3/8)\to(x, y)$ (since $x_k\to 0$). Furthermore, since $\|y\|=1=\sup|y_i|+(\sum y_i^2/2^i)^{1/2}\leqq 2\sup|y_i|$, we have $\sup|y_i|\geqq 1/2$. Hence

$$\sup|y_i^k|=\max(3/8,\sup_{i\neq k}|y_i|)=\sup|y_i|.$$

Thus, $\|y^k\|=\sup|y_i|+[\sum y_i^2/2^i-(y_k^2-9/16)/2^k]^{1/2}\to\|y\|$. Finally, $\|y^k-y\|\geqq\sup_i|y_i^k-y_i|=|3/8-y_i|>1/8$. Turning now to the other case, suppose there exists $\epsilon>0$ and a subsequence $\{y_{i_k}\}$ of $\{y_i\}$ such that $|y_{i_k}|\geqq 2\epsilon$, $k=1, 2, 3, \cdots$. Define $y^k\in E^*$ by $y_i^k=y_i$ if $i\neq i_k$, $y_{i_k}^k=\epsilon$. Then $(x, y^k)\to(x, y)$ as before, and $\sup|y_i-y_i^k|=|y_{i_k}-\epsilon|\geqq\epsilon$. Furthermore, $\|y^k\|\to\|y\|$, which completes the proof of (iii).

To prove (iv), use the fact from (i) that sm $S=S$, so $P(E)\cap S^*\subset\exp U^*$, and apply (ii). To prove (v), we can use Theorem 4.3(iii) and the fact that E (isomorphic to l_1) is separable, while E^* (isomorphic to m) is not.

Parts (ii), (iv) and (v) of the above example show that the necessary conditions (subreflexivity and density of ext U^* in S^*) given by Theorem 4.3 (i) are not sufficient. We know of no example, however, showing that the sufficient condition (str S^* dense in S^*) of Theorem 2.2 is not necessary. While we believe such an example exists, it seems difficult to obtain.

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INSTITUTE FOR ADVANCED STUDY

ISOMETRIES OF GROUP ALGEBRAS[1]

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Let G be a locally compact abelian group, $\hat{G}$ its character group, and A the group algebra of G. Associated with any automorphism ϕ of A [2] is a homeomorphism τ of $\hat{G}$ onto itself, with the property that, for $\alpha \in \hat{G}$, $f \in A$ and $\mathfrak{F}$ the Fourier transform, $\mathfrak{F}(\phi(f))(\tau\alpha) = \mathfrak{F}(f)(\alpha)$. Results of Helson [2] and Wendel [4] state that if e is the unit of $\hat{G}$, then

$$\tau(e)\tau(xy) = \tau(x)\tau(y), \qquad \text{for all } x, y \in \hat{G}, \tag{1}$$

if and only if ϕ is an isometry. The object of the present note is to give a further equivalent form of the statement that ϕ is an isometry.

Let T_α, $\alpha \in \hat{G}$, be that operator on A which for all $x \in G, f \in A$ satisfies $(T_\alpha f)(x) = f(x)(x, \alpha)$. We consider homomorphisms ϕ of A onto A such that to each α there is a $\rho(\alpha) \in \hat{G}$ such that

$$\phi T_\alpha = T_{\rho(\alpha)}\phi, \qquad \alpha \in \hat{G}. \tag{2}$$

Our result is that such homomorphisms are isomorphisms and indeed isometries.

THEOREM 1. *Let ϕ be a homomorphism of the group algebra A of the locally compact abelian group G onto itself. Suppose that ϕ satisfies* (2); *then ϕ is an isomorphism.*

Let K be the kernel of ϕ. Since ϕ is automatically continuous [3], K is a proper closed ideal of A. The Wiener Tauberian theorem thus yields a maximal regular ideal M containing K. From condition (2) it follows that for $k \in K$ and $\beta \in \hat{G}$, $\phi T_\beta(k) = T_{\rho(\beta)}\phi(k) = 0$, and there-

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