A REPRESENTATION THEOREM FOR BOUNDED
CONVEX SETS

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1. Introduction. It is well known that any closed convex subset of
a normed linear space can be represented as the intersection of all
those closed half-spaces which contain it. It is also well known (and
easy to prove) that if the closed convex set is bounded, and if the space
under consideration is Euclidean n-space, then the set can be repre-
sented as the intersection of cells, i.e., the following is true in the case
when E is Euclidean n-space:

If C is a bounded closed convex subset of the normed linear space E,
and if x∈E∩C, then there exist y∈E and r>0 such that the cell
N₁y = {z: \|z - y\| ≤ r} contains C but not x.

In the present paper we consider those spaces E for which the above
statement holds for all such C; such a space is said to have the property (a): Every bounded closed convex set C can be represented as the
intersection of cells.

Our study was motivated in part by a paper of S. Mazur [5], in
which property (a) was shown to hold for all reflexive Banach spaces
having a strongly differentiable norm, and in which the following
criterion for weak sequential convergence was proved:

Suppose E has property (a). Then a sequence \{xₙ\} in E converges
weakly to the point x if and only if (i) the sequence \{xₙ\} is bounded and
(ii) every cell containing infinitely many xₙ also contains x.

In what follows we obtain a general sufficient condition for prop-
erty (a), as well as some necessary conditions, these conditions all be-
ing given in the form of density criteria for certain subsets of U*, the
unit cell of the conjugate space E*. For finite dimensional E, these
conditions are equivalent, and we obtain the following interesting
result:

A finite dimensional space E has property (a) if and only if the set
of extreme points of U* is dense in the boundary of U*.

An example is given which shows that in infinite dimensional
spaces, the above condition (while necessary) does not imply property
(a). The same example provides an answer to a question of Mazur
[5], by showing the existence of a Banach space in which the norm is

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weakly differentiable everywhere (except at the origin) but strongly
differentiable nowhere.

Before proceeding, we list some definitions used throughout the
paper. Denote the set of real numbers by \( \mathbb{R} \). Let \( S \) and \( S^* \) denote the
unit spheres of \( E \) and \( E^* \) respectively, i.e., those elements of norm 1,
and let \( U \) and \( U^* \) denote the unit cells of \( E \) and \( E^* \), i.e., those ele-
ments of norm at most 1.

2. A sufficient condition for property (\( \varnothing \)). If \( C \) is a bounded, closed,
convex subset of a normed linear space \( E \) such that \( \phi \in C \), let \( C' \) be
the set of all linear functionals \( f \) in \( E^* \) such that

\[
\inf f(C) = \inf \{ f(x) : x \in C \}
\]

is positive. It follows from the basic separation theorem \([1]\) that \( C' \)
is nonempty. A subset \( K \) of a linear space is called a convex cone if \( K \)
is convex and if \( x \in K \) and \( \lambda > 0 \) imply \( \lambda x \in K \).

**Lemma 2.1.** If \( C \) is a bounded, closed convex subset of a normed linear
space \( E \), then \( C' \) is an open convex cone in \( E^* \).

**Proof.** It is easily verified that \( C' \) is a convex cone. To see that \( C' \)
is open, suppose that \( f \in C' \), let \( a = \inf \{ f(x) : x \in C \} > 0 \), and choose \( \varepsilon > 0 \) such that \( a + \varepsilon \) is a convex cone in \( E^* \).

We say that the norm in \( E \) is strongly differentiable at the point
\( x \neq \phi \) if there exists an \( f \) in \( S^* \) such that \( \lim_{y \rightarrow x} \delta_\varepsilon(y) = 0 \), where

\[
\delta_\varepsilon(y) = \|x + y\| - \|x\| - \varepsilon f(y).
\]

Let \( \text{str} \ S^* \) be the set of all \( f \) in \( S^* \) which
attain their norm at a point of strong differentiability, i.e., for which
there exists an \( x \) in \( S \) such that the norm in \( E \) is strongly differentiable
at \( x \) and \( f(x) = 1 \). The following theorem, while more general than
Mazur's theorem \([5]\), is proved in much the same way.

**Theorem 2.2.** A normed linear space \( E \) has property (\( \varnothing \)) if \( \text{str} \ S^* \) is
dense in \( S^* \).

**Proof.** Suppose that \( C \) is a bounded, closed convex subset of \( E \)
and that \( y \in E \sim C \). We may assume, without loss of generality, that
\( y = \phi \). Since \( \text{str} \ S^* \) is dense in \( S^* \) and (by Lemma 2.1) \( C' \cap S^* \) is open
in \( S^* \), there exists an \( f \) in \( \text{str} \ S^* \) and an \( x \) in \( S \) such that \( \inf f(C) > 0 \),
\( f(x) = 1 \), and the norm in \( E \) is strongly differentiable at \( x \).

Choose \( \varepsilon > 0 \) such that \( \inf f(C) > 2 \varepsilon > 0 \) and let \( \varepsilon = \varepsilon x \). For each \( r > 1 \) let \( N_r \) be the cell of radius \( \|(r - 1)\varepsilon\| = (r - 1)\varepsilon \) which is centered at \( rz \). The
origin is not in \( N_r \) for any \( r > 1 \), so it will suffice to show that \( C \subset N_r \)
for some \( r > 1 \). Suppose not; then there exist sequences \( \{r_n\} \) and \( \{x_n\} \).
such that \( r_n > 1, r_n \to \infty, \|x_n - r_n z\| > (r_n - 1)e \) and \( x_n \in C \) for each \( n \). The sequence \( \{x_n\} \) is bounded because \( C \) is bounded, so we have \( y_n = -x_n/er_n \to \phi \) as \( n \to \infty \). Hence, by strong differentiability of the norm at \( x \), we must have \( \delta_{\phi}(y_n)/\|y_n\| \to 0 \) as \( n \to \infty \). But \( er_n \delta_{\phi}(y_n) = \|r_n z - x_n\| - er_n + f(x_n) > (r_n - 1)e - er_n + 2e = e \). Hence
\[
\delta_{\phi}(y_n)/\|y_n\| = er_n \delta_{\phi}(y_n)/\|x_n\| > e/\|x_n\|,
\]
a contradiction, since the \( x_n \) are bounded.

If \( E \) is a normed linear space let \( P(E) \) be the set of all those linear functionals \( f \) in \( E^* \) which attain their supremum on \( S \), i.e. those \( f \) for which there exists \( x \) in \( S \) such that \( f(x) = \|f\| \). We call \( E \) subreflexive if \( P(E) \) is norm-dense in \( E^* \), or equivalently, if \( P(E) \cap S^* \) is dense in \( S^* \). Note that, always, \( S^* \subset P(E) \cap S^* \), so it follows immediately from our hypothesis in Theorem 2.2 that \( E \) is subreflexive. In Theorem 4.3 (i) it will be shown that subreflexivity is a necessary condition of property \( (d) \). (There exist non-subreflexive normed linear spaces \( [6] \); see also \( [7] \) but, to the best of our knowledge, the question "Is every Banach space subreflexive?" remains open.)

3. An elementary lemma. In this section we prove a lemma which describes a basic connection between pairs of linear functionals and the hyperplanes determined by them. Roughly speaking, the lemma asserts the intuitively obvious fact that if two functionals \( f \) and \( g \) of norm one have their hyperplanes \( f^{-1}(0) \) and \( g^{-1}(0) \) sufficiently close together, then one of \( \|f - g\|, \|f + g\| \) must be small.

**Lemma 3.1.** Suppose that \( E \) is a normed linear space and that \( e > 0 \). If \( f, g \in S^* \) are such that \( f^{-1}(0) \cap U \subset g^{-1}[-e/2, e/2] \), then either \( \|f - g\| \leq e \) or \( \|f + g\| \leq e \).

**Proof.** By the Hahn-Banach theorem we can choose \( h \in E^* \) such that \( h = g \) on \( f^{-1}(0) \) and \( \|h\| = \sup \{g(U \cap f^{-1}(0))\} \). Then, by hypothesis, \( \|h\| \leq e/2 \). Furthermore, since \( g - h \) vanishes on \( f^{-1}(0) \) there exists \( \alpha \) in \( R \) such that \( g - h = \alpha f \). Hence \( \|g - \alpha f\| = \|h\| \leq e/2 \). Assuming \( \alpha \geq 0 \), we will show that \( \|f - g\| \leq e \). (Otherwise, the same proof applied to \( (-\alpha)f \) would show that \( \|f + g\| \leq e \).) If \( \alpha \geq 1 \), then \( \alpha^{-1} \leq 1 \) and \( \|g - f\| = \|(1 - \alpha^{-1})g + \alpha^{-1}(g - \alpha f)\| \leq 1 - \alpha^{-1} + \alpha^{-1}\|g - \alpha f\| \). Also, \( \alpha = \|\alpha f\| \leq \|g - \alpha f\| \) so \( 1 - \alpha^{-1} \leq (1 + \|g - \alpha f\|)^{-1}\|g - \alpha f\| \leq \|g - \alpha f\| \). Hence \( \|g - f\| \leq 2\|g - \alpha f\| \leq e \). If \( 0 \leq \alpha < 1 \), then \( \|g - f\| \leq \|g - \alpha f\| + \|((1 - \alpha)f\| = \|g - \alpha f\| + 1 - \alpha = \|g - \alpha f\| + \|g\| - \|\alpha f\| \leq 2\|g - \alpha f\| \leq e \), which completes the proof.


\(^2\) It should be noted that this lemma provides a short proof of Theorem 1.2 of \([6]\).
4. Some consequences of property (g). If \( x \) is in \( S \), let \( F_x \) be the face of \( S^* \) determined by \( x \), i.e., \( F_x = \{ f : \| f \| = 1 = f(x) \} \). Each such face is closed, convex and (by the Hahn-Banach theorem) nonempty. The next lemma is basic to the rest of the paper.

**Lemma 4.1.** Suppose that the normed linear space \( E \) has property (g). If \( f \in S^* \) and \( 0 < \epsilon < 1 \) there exist \( \delta(\epsilon) > 0 \) and \( x \in S \) such that \( y \in S \cap N_{\delta} x \) implies \( F_y \subset N_{\delta} f \).

**Proof.** If \( f \in S^* \) and \( 0 < \epsilon < 1 \) let \( D = U \cap f^{-1}(0) \) and pick \( u \in S \) such that \( f(u) > \epsilon \). Since \( D \) is bounded, closed and convex, and \( u' \in D \), property (g) implies the existence of \( r > 0 \) and \( z \in E \) such that \( u' \in N_{\delta} z \subset D \). Let \( w \) be the intersection of the segment \([u', z]\) with the boundary of \( N_{\delta} z \); then if \( C \) denotes the convex hull of \( u' \) and \( N_{\delta} z \) we can conclude that \( w \in \text{int} \ C \), the interior of \( C \). Hence there exists \( \alpha > 0 \) such that \( \| v - w \| \leq \alpha \) implies \( v \in \text{int} \ C \). Note that for any such \( v \) there exists \( v' \in \text{int} \ N_{\delta} z \) and \( \lambda \in [0, 1] \) such that \( v = \lambda u' + (1 - \lambda) v' \).

Now let \( x = r^{-1}(w - z) \) and let \( \delta = \arctan \). If \( y \in S \cap N_{\delta} x \) and \( g \in F_y \) then \( g(y) = \| g \| = 1 = \sup g(U) \). Letting \( v = ry + z \) we have \( g(v) = r \sup g(U) + g(z) = \sup g(N_{\delta} z) \). Now \( \| v - w \| = \| v - (rx + z) \| = r \| x - y \| \leq \delta = \alpha \), so there exist \( v' \) and \( \lambda > 0 \) as above. Since \( v' \in \text{int} \ N_{\delta} z \), we have \( g(v') < g(v) \), while \( g(\phi) \in D \subset N_{\delta} z \) implies \( g(\phi) \geq 0 \). Hence \( g(v) = \lambda g(u') + (1 - \lambda) g(v') < \lambda g(u') + (1 - \lambda) g(v) \) or \( 0 \leq g(v) < g(u') \). Since \( g(u') \leq \| g \| \cdot \| u' \| = 1, g(u') = (\epsilon/2) g(u) \leq \epsilon/2 \) and therefore \( g(v) \leq \epsilon/2 \). Thus, for any \( w' \in D \subset N_{\delta} z \), \( g(w') \leq g(v) < \epsilon/2 \) and, by the symmetry of \( D \) about \( \phi \), \( g(w') > -\epsilon/2 \). Hence \( D = f^{-1}(0) \cap U \subset g^{-1}[-\epsilon/2, \epsilon/2] \) and therefore, by Lemma 3.1, either \( \| f - g \| \leq \epsilon \) or \( \| f + g \| \leq \epsilon \). But \( g(u) > (2/\epsilon) g(v) \geq 0 \) and \( f(u) > \epsilon \), so \( \epsilon < (f + g)(u) \leq \| f + g \| \), which shows that \( \| f - g \| \leq \epsilon \). Since this is true for any \( g \in F_y \), we have \( F_y \subset N_{\delta} f \).

If \( K \) is a convex subset of a linear space we say that a subset \( A \) of \( K \) is an extremal subset of \( K \) if \( A \) is nonempty and both \( A \) and \( K - A \) are convex. An extremal subset of \( K \) which consists of a single point is called an extreme point of \( K \). It is easy to verify that if \( x \in S \) then \( F_x \) is an extremal subset of \( U^* \), the unit cell of \( E^* \). Furthermore, the set \( \{ f : f(x) = 1 \} \) is weak*-closed in \( E^* \) and hence, since \( U^* \) is weak*-compact, \( F_x = U^* \cap \{ f : f(x) = 1 \} \) is weak*-compact. It follows from the Krein-Milman theorem [1] that \( F_x \) has at least one extreme point, and (using the fact that \( F_x \) is extremal in \( U^* \)) this point is also an extreme point of \( U^* \). If \( B \) is a convex set, let \( \text{ext} \ B \) be the set of all extreme points of \( B \).

Let \( \phi \) be a "selection" for the mapping \( x \to F_x \) such that, for each \( x \in S \), \( \phi(x) \) is an extreme point of \( F_x \). If \( A \) is any subset of \( S \), then the cardinality of \( \phi(A) = \{ \phi(x) : x \in A \} \) is no greater than that of \( A \). We have the following immediate (and useful) corollary to Lemma 4.1.
Corollary 4.2. If $E$ is a normed space having property $(\mathcal{G})$, and if $A$ is dense in $S$, then $\varphi(A)$ is dense in $S^*$.

We say that $x \in S$ is a smooth point of $S$ if $F_x$ consists of exactly one point. Denote the set of smooth points of $S$ by $\text{sm } S$. A point $y$ of a convex set $K \subset E$ is called an exposed point of $K$ if there exists a hyperplane $H$ supporting $K$ at $y$ such that $H \cap K = \{y\}$. Equivalently, $y \in \text{exp } K$ (the set of all exposed points of $K$) if and only if there exists a linear functional $f \in E^*$ such that $\sup f(K) = f(y)$ and $f(x) < f(y)$ for $x \in K \setminus \{y\}$. Note that, always, $\text{exp } K \subset \text{ext } K$. If $x \in \text{sm } S$ and $f$ is the unique point of $F_x$, consider $x$ as a continuous linear functional on $E^*$ (by letting $x(g) = g(x)$ for each $g \in E^*$); then if $g \in U^* \setminus \{f\}$, $x(g) = 1 = x(f)$, so $f \in \text{exp } U^*$. Thus, the following inclusions always hold: $\varphi(\text{sm } S) \subset \text{exp } U^* \subset \text{ext } U^*$.

**Theorem 4.3.** Suppose that $E$ is a normed linear space having property $(\mathcal{G})$; then the following implications are valid:

(i) The set $\varphi(S)(\subset P(E)\cap \text{ext } U^*)$ is dense in $S^*$; hence $E$ is subreflexive and $\text{ext } U^*$ is dense in $S^*$.

(ii) If $\text{sm } S$ is dense in $S$, then $\varphi(\text{sm } S)(\subset \text{exp } U^*)$ is dense in $S^*$.

(iii) If $E$ is separable, then $\text{exp } U^*$ is separable.

(iv) If $E$ is separable and complete, then $\text{exp } U^*$ is dense in $S^*$.

**Proof.** All the statements, with the exception of (iv), follow trivially from Corollary 4.2 and the preceding remarks and definitions. To prove (iv), we use the theorem of Mazur [4] which states that, for separable Banach spaces, $\text{sm } S$ is dense in $S$; (ii) now yields (iv).

**Theorem 4.4.** If $E$ is a finite dimensional normed linear space the following assertions are equivalent:

(i) The space $E$ has property $(\mathcal{G})$.

(ii) The set $\text{exp } U^*$ is dense in $S^*$.

(iii) The set $\text{ext } U^*$ is dense in $S^*$.

**Proof.** That (i) implies (iii) follows from Theorem 4.3 (i). By a theorem of Klee [3, Theorem 2.3], $\text{exp } U^*$ is dense in $\text{ext } U^*$, so (iii) implies (ii). To show that (ii) implies (i), we first show that $\text{exp } U^* = \text{str } S^*$ in any finite dimensional space, then we apply Theorem 2.2. Mazur has pointed out [5, p. 130] that, in a finite dimensional space $E$, a point $x$ is in $\text{sm } S$ if and only if the norm in $E$ is strongly differentiable at $x$. Thus, if $f \in \text{str } S^*$, $f$ attains its supremum at a smooth point of $S$ and therefore, using the remarks preceding Theorem 4.3, $f \in \text{exp } U^*$. On the other hand, if $f \in \text{exp } U^*$, there exists $X$ in $E^{**}$ such that $X(f) = 1 = \|X\|$ and $X(g) < 1$ for $g \in U^*$.
\[ \sim \{f\} \] From the reflexivity of finite dimensional spaces it follows that there exists an \( x \in S \) such that \( g(x) = X(g) \) for each \( g \in E^* \). From the above conditions on \( X \) we conclude that \( F_x \) is the single point \( f \), i.e., \( x \in \text{sm } S \). Thus, the norm in \( E \) is strongly differentiable at \( x \) and therefore \( f \in \text{str } S^* \), which was to be shown.

If \( E \) is two-dimensional it is easy to verify that \( \text{ext } U^* \) is dense in \( S^* \) if and only if \( \text{ext } U^* = S^* \), i.e., if and only if \( E^* \) is strictly convex. This property is equivalent (see e.g. [6]) to the property that \( E \) itself is smooth, i.e., that \( \text{sm } S = S \). Thus we have the following corollary to Theorem 3.5.

**Corollary 3.6.** A two-dimensional normed linear space \( E \) has property (\( s \)) if and only if \( E \) is smooth.

5. **An example.** In this section we exhibit a separable Banach space \( E \) which is isomorphic to \( l_1 \) (the space of absolutely summable sequences) and which has the following properties:

(i) \( E \) is smooth and \( E^* \) is strictly convex.

(ii) \( E \) is subreflexive.\(^1\)

(iii) The set \( \text{str } S^* \) is empty, hence the norm in \( E \) is not strongly differentiable at any point of \( S \).

(iv) The set \( \text{exp } U^* \) is dense in \( S^* \).

(v) \( E \) does not have property (\( s \)).

Now, smoothness of \( E \) is equivalent to weak differentiability of the norm at each \( x \neq \phi \) in \( E \) (see, e.g. [1]), hence properties (i) and (iii) give a negative answer to the question, noted by Mazur in [5], as to whether weak differentiability always implies strong differentiability of the norm.

We will obtain the space \( E \) by using a theorem of Day's [2, Theorem 5] to renorm \( l_1 \). We first define a new norm for \( m = l_1^* \) (the space of bounded sequences \( y = \{y_i\} \) by \( \|y\| = \sup |y_i| + (\sum y_i^2)^{1/2} \). If \( x = \{x_i\} \) is in \( l_1 \), let \( \|x\| = \sup \{ \sum x_i y_i : y = \{y_i\} \in m \text{ and } \|y\| = 1 \} \). [To see how Day's theorem is applied to yield these norms, let the space \( B_0 \) of his theorem be \( l_2 \), let \( B \) be \( l_1 \) and let \( T \) be the mapping from \( l_2 \) into \( l_1 \) defined by \( T(\{x_i\}) = \{x_i/2^{1/2}\} \).] Then, letting \( E \) be \( l_1 \) so renormed, we can conclude from Day's theorem that \( E \) is isomorphic with \( l_1 \), that \( E^* \) is the space of bounded sequences with the norm described above, that \( E \) is smooth and that \( E^* \) is strictly convex. Thus, \( E \) has property (i) above.

To prove that \( \text{str } S^* \) is empty, we first prove the following simple lemma:

If the norm in a linear space \( E \) is strongly differentiable at the point \( x \neq \phi \), if \( f \in S^* \) is such that \( f(x) = \|x\| \), if \( \{f_n\} \subset B^* \) is such that
$f_n(x) \to f(x)$ and $\|f_n\| \to \|f\| = 1$, then $\|f_n - f\| \to 0$. To see this, note first that we can assume $\|f_n\| = 1$ for all but a finite number of $n$, since the lemma is true for the sequence $\{f_n\}$ if and only if it is true for the sequence $\{f_n/\|f_n\| : \|f_n\| \neq 0\}$. Now if $\|f_n - f\| \to 0$ we can choose (taking a subsequence if necessary) $\varepsilon > 0$ and $y_n \in S$ such that $(f_n - f)(y_n) \geq 2\varepsilon$. Now, let $x_n = e^{-1} (\|x\| - f_n(x)) y_n$ and use the fact that $\|x + x_n\| \geq f_n(x) + f_n(x_n)$ to show that $\|x_n\|^{-1} (\|x + x_n\| - \|x\| - f(x_n)) \geq \varepsilon$, contradicting the assumption of strong differentiability at $x$ and the fact that $\|x_n\| \to 0$.

Thus, in order to show that no point $x \neq \phi$ in $E$ is a point of strong differentiability, it suffices to show that for each $x \neq \phi$ in $E$ and any $y \in S^*$ such that $(x, y) = \|x\|$ there exists a sequence $\{y^k\} \subset E^*$ such that $(x, y_k) \to \|x\|$, $\|y^k\| \to \|y\|$, but $\|y^k - y\| \to 0$. (The latter is equivalent, of course, to $\lim_{k \to \infty} \sup_i |y^k_i - y_i| \neq 0$.) Assuming we have such $x$ and $y$, we consider two cases. First, suppose that $y_i \to 0$. Pick an integer $n$ such that $|y_i| < 1/4$ if $i > n$ and define $y^k \in E^*(k > n)$ by $y^k_i = y_i$ if $i \neq k$, $y^k_k = 3/8$. Then $(x, y^k) = (x, y) - x_k(y_k - 3/8) \to (x, y)$ (since $x_k \to 0$). Furthermore, since $\|y\| = 1 = \sup_i |y_i| + (\sum y^2_i/2)^{1/2} \leq 2\sup_i |y_i|$, we have $\sup_i |y_i| \geq 1/2$. Hence

$$\sup_i |y^k_i| = \max(3/8, \sup_i |y_i|) = \sup_i |y_i|.$$  

Thus, $\|y^k\| = \sup_i |y_i| + (\sum y^2_i/2^i - (y^2_k - 9/16)/2^k)^{1/2} \to \|y\|$.

Finally, $\|y^k - y\| \geq \sup_i |y^k_i - y_i| = |3/8 - y_i| > 1/8$. Turning now to the other case, suppose there exists $\varepsilon > 0$ and a subsequence $\{y_{i_k}\}$ of $\{y_i\}$ such that $|y_{i_k}| \geq 2\varepsilon$, $k = 1, 2, 3, \ldots$. Define $y^k \in E^*$ by $y^k_i = y_i$ if $i \neq i_k$, $y^k_{i_k} = \varepsilon$. Then $(x, y^k) \to (x, y)$ as before, and $\sup_i |y_i - y^k_i| = |y_{i_k} - \varepsilon| \geq \varepsilon$. Furthermore, $\|y^k\| \to \|y\|$, which completes the proof of (iii).

To prove (iv), use the fact from (i) that $m_S = S$, so $P(E) \cap S^* \subset \operatorname{exp} U^*$, and apply (ii). To prove (v), we can use Theorem 4.3(iii) and the fact that $E$ (isomorphic to $l_1$) is separable, while $E^*$ (isomorphic to $m$) is not.

Parts (ii), (iv) and (v) of the above example show that the necessary conditions (subreflexivity and density of $\operatorname{ext} U^*$ in $S^*$) given by Theorem 4.3 (i) are not sufficient. We know of no example, however, showing that the sufficient condition (str $S^*$ dense in $S^*$) of Theorem 2.2 is not necessary. While we believe such an example exists, it seems difficult to obtain.

**BIBLIOGRAPHY**


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__ISOMETRIES OF GROUP ALGEBRASI__

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Let $G$ be a locally compact abelian group, $\hat{G}$ its character group, and $A$ the group algebra of $G$. Associated with any automorphism $\phi$ of $A$ [2] is a homeomorphism $\tau$ of $\hat{G}$ onto itself, with the property that, for $\alpha \in \hat{G}$, $f \in A$ and $\hat{f}$ the Fourier transform, $\hat{f}(\phi(f))(\tau \alpha) = \hat{\phi}(f)(\alpha)$. Results of Helson [2] and Wendel [4] state that if $e$ is the unit of $\hat{G}$, then

\[(1) \quad \tau(e) \tau(xy) = \tau(x) \tau(y), \quad \text{for all } x, y \in \hat{G},\]

if and only if $\phi$ is an isometry. The object of the present note is to give a further equivalent form of the statement that $\phi$ is an isometry.

Let $T_\alpha$, $\alpha \in \hat{G}$, be that operator on $A$ which for all $x \in G$, $f \in A$ satisfies $(T_\alpha f)(x) = f(x)(x, \alpha)$. We consider homomorphisms $\phi$ of $A$ onto $A$ such that to each $\alpha$ there is a $\rho(\alpha) \in \hat{G}$ such that

\[(2) \quad \phi T_\alpha = T_{\rho(\alpha)} \phi, \quad \alpha \in \hat{G}.\]

Our result is that such homomorphisms are isomorphisms and indeed isometries.

**Theorem 1.** Let $\phi$ be a homomorphism of the group algebra $A$ of the locally compact abelian group $G$ onto itself. Suppose that $\phi$ satisfies (2); then $\phi$ is an isomorphism.

Let $K$ be the kernel of $\phi$. Since $\phi$ is automatically continuous [3], $K$ is a proper closed ideal of $A$. The Wiener Tauberian theorem thus yields a maximal regular ideal $M$ containing $K$. From condition (2) it follows that for $k \in K$ and $\beta \in \hat{G}$, $\phi T_\beta(k) = T_{\rho(\beta)} \phi(k) = 0$, and there-

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