THE NONTRIVIALITY OF THE RESTRICTION MAP IN THE COHOMOLOGY OF GROUPS

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An unpublished result[2] of B. Mazur states that if π is any nontrivial finite group then there is an $i>0$ such that $H^i(\pi, Z)\neq 0$. It is, course, trivial that $H^i(\pi, A)\neq 0$ for some π-module A. The point of Mazur's theorem is that we can even take $A=Z$, the ring of integers with trivial π-action. Mazur's proof of this theorem is geometric. It involves imbedding π in a compact Lie group G and studying the Leray-Cartan spectral sequence of the covering $G\rightarrow G/\pi$.

The purpose of this paper is to prove the following theorem which generalizes Mazur's result.[3]

THEOREM 1. *Let π be a finite group and ρ a nontrivial subgroup of π. Then the restriction map $i(\rho, \pi)\colon H^i(\pi, Z)\rightarrow H^i(\rho, Z)$* [2, Chapter XII, §8] *is nonzero for an infinite number of values of $i>0$.*

As a consequence of this theorem, we get a generalization of Mazur's result.

COROLLARY 1. *Let π be a finite group and let p be a prime dividing the order of π. Then $H^i(\pi, Z)$ has a nonzero p-primary component for an infinite number of values of $i>0$.*

To see this we have merely to use Theorem 1, choosing for ρ any nontrivial p-group in π.

The proof of Theorem 1 will also be geometric. In fact, I will actually prove the following much more general theorem whose proof must necessarily be geometric.

THEOREM 2. *Let G be a compact, not necessarily connected Lie group. Let H be a closed nontrivial subgroup of G, also not necessarily connected. Let $f\colon B_H\rightarrow B_G$ be the map of classifying spaces induced by the inclusion map $H\rightarrow G$* [1, §1]. *Then $f^*\colon H^i(B_G, Z)\rightarrow H^i(B_H, Z)$ is nonzero for an infinite number of values of i.*

REMARK. If H has an element of order p, the proof of this theorem will also show that $f^*\colon H^i(B_G, Z_p)\rightarrow H^i(B_H, Z_p)$ is nontrivial for an infinite number of values of i. If H is infinite, it will show that

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[3] This theorem was suggested to me by a problem of J. T. Tate.

$f^*: H^i(B_G, Q) \to H^i(B_H, Q)$ is nonzero for an infinite number of values of i. Here Q is the field of rational numbers.

PROOF. By the Peter-Weyl theorem G has a faithful unitary representation [4, Chapter VI, Theorem 4] and so can be imbedded in a unitary group $U(n)$. Also, H has a subgroup isomorphic to Z_p for some prime p. This is trivial if H is finite, but if H is infinite it contains a torus [3, Exposé 23, Theorem 1] which clearly has a subgroup isomorphic to Z_p. Since the map $B_{Z_p} \to B_{U(n)}$ factors through f, it will be sufficient to prove the theorem for the case $H \approx Z_p$ and $G \approx U(l)$. (If H is infinite and we are trying to show that $H^i(B_G, Q) \to H^i(B_H, Q)$ is nontrivial, it will suffice to consider the case where $G \approx U(n)$ and H is a circle group. The rest of the proof will be substantially the same.)

Assume then that $H \approx Z_p$, $G \approx U(l)$. Imbed H in a maximal torus T of G. This can be done by taking any maximal torus T containing a generator of H [3, Exposé 23, Theorem 1]. Now, $H^*(B_T, Z)$ is a polynomial ring over Z with generators $t_1, \cdots, t_l \in H^2(B_T, Z)$. The image of $H^*(B_G, Z)$ in $H^2(B_T, Z)$ consists of all symmetric polynomials in $t_1, \cdots, t_l$ [1, §4]. Therefore to prove the theorem it will be sufficient to find sufficiently many symmetric polynomials which map nontrivially into $H^*(B_H, Z)$ under the map g^* induced by $g: B_H \to B_T$. This map g is, of course, induced by the inclusion $H \to T$.

Now, $H^*(B_H, Z)$ is a polynomial ring over Z_p with a single generotor $\alpha \in H^2(B_H, Z)$ [2, Chapter XII, §7]. Therefore $g^*(t_\nu) = r_\nu \alpha$ with $r_\nu \in Z_p$. I claim that at least one $r_\nu \neq 0$. Suppose to the contrary that all $r_\nu = 0$. Then $g^*: H^2(B_T, Z) \to H^2(B_H, Z)$ must be zero. Now $g: B_H \to B_T$ is a fiber map with fiber T/H [1, §1]. Of course, T/H is a torus, being a connected abelian Lie group. The map $g^*: H^2(B_T, Z) \to H^2(B_H, Z)$ is just the map $E_2^{2,0} \to E_\infty^{2,0}$ in the spectral sequence of this fibration. If it is zero, all elements of $E_2^{2,0}$ must bound. Therefore $d_2: E_2^{0,1} \to E_2^{2,0}$ must be onto. This shows that T/H has rank l and that $H_1(T/H, Z) = E_2^{0,1}$ has a base $\{x_\nu\}$ such that $d_2 x_\nu = t_\nu$. (Of course it is trivial that T/H has rank l, H being finite, but I have arranged the proof so that it works for $H = S^1$ without essential change.) Now $E_2^{0,2} = H^2(T/H, Z)$ has a base $x_\mu x_\nu$ with $\mu < \nu$. Since d_2 is a derivation, $d_2(x_\mu x_\nu) = t_\mu \otimes x_\nu - t_\nu \otimes x_\mu$ in $E_2^{2,1} = H^2(B_T) \otimes H^1(T/H)$. Since these elements are linearly independent in $E_2^{2,1}$, d_2 is a monomorphism on $E_2^{0,2}$ and so $E_3^{0,2} = 0$. Also $E_3^{2,0} = 0$ and $E_2^{1,1} = 0$. Thus the spectral sequence shows that $H^2(B_H, Z) = 0$ which is absurd.

Now let s be the number of indices ν for which $r_\nu \neq 0$. By renumbering we can assume that $r_\nu \neq 0$ for $\nu = 1, 2, \cdots, s$ and $r_\nu = 0$ for $\nu > s$.

Let x be the sth elementary symmetric function in $t_1, \cdots, t_l$. Then, for $k>0$,

$$g^*(x^k) = \left(\prod_1^s r_\nu\right)^k \alpha^{sk} \neq 0.$$

Since the x^k are symmetric polynomials and have arbitrarily large dimensions, this proves the theorem.

REMARK. If l is the smallest dimension of a faithful representation of G over the complex numbers, the proof shows that $f^*: H^i(B_G, Z) \to H^i(B_H, Z)$ is nonzero for some $i \leqq 2l$ (since $i=2s$ and $s \leqq l$). This is a best possible result if no further conditions are placed on G, H and l. To see this for finite groups, let H be the cyclic group of order p permuting p symbols and let G be the normalizer of H in the symmetric group S_p.

If R denotes the real numbers, duality shows that $f_*: H_i(B_H, R/Z) \to H_i(B_G, R/Z)$ is nonzero for an infinite number of values of i. But, if π is finite, $H_i(\pi, R/Z) \approx H_{i-1}(\pi, Z)$, cf. [**2**, Chapter XII, Proof of Theorem 6.6]. Therefore Theorem 1 has the following corollary.

COROLLARY 2. *Let π be a finite group and ρ a nontrivial subgroup of π. Then the induced map $H_i(\rho, Z) \to H_i(\pi, Z)$ is nontrivial for an infinite number of values of $i>0$.*

Equivalently, we may say that the transfer $t(\pi, \rho): \hat{H}^i(\rho, Z) \to \hat{H}^i(\pi, Z)$ is nonzero for an infinite number of negative values of i [**2**, Chapter XII, Exercise 8].

Note that the example $Z_p \subset Z_p + Z_p$ shows that the restriction map can be zero in all negative dimensions and the transfer zero in all positive dimensions.

It would be interesting to have a purely algebraic proof of Theorem 1 but I know of no such proof.

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