A COUNTEREXAMPLE OF KOEBE'S FOR SLIT MAPPINGS[1]

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1. We refer to a region Ω of the extended z-plane as a (parallel) slit domain if $\infty \in \Omega$, and if the components of the boundary, $\partial\Omega$, are either points, or segments ("slits") parallel to a common line, which without loss of generality will be assumed to be the y-axis ($z = x + iy$). It was originally conjectured by Koebe that if two slit domains Ω_1 and Ω_2 are conformally equivalent, that is, if there exists a function f, schlicht in Ω_2, such that $f(\infty) = \infty$, $f(\Omega_2) = \Omega_1$, then, unless f is linear, at most one of the two sets, $E_1 = \partial\Omega_1$, $E_2 = \partial\Omega_2$, has area zero. Later on Koebe [5] outlined the construction of a counterexample in which (using the present notation)

(a) the components of E_1 are not all points,

(b) the projection of E_1 onto the x-axis has linear Lebesgue measure zero,

(c) E_2 is a compact, totally disconnected subset of the x-axis.

Although Koebe's example, and variants therefore, have been applied repeatedly in connection with various counterexamples in complex variable theory (see, for instance, [7]) it does not appear to have been previously noted in the literature that the reasoning in [5] contains a gap. The statement containing the word "offenbar" in the last paragraph of page 62 of [5] is incorrect. If P denotes the intersection of Koebe's S_1 with a line parallel to the y-axis, then P is denumerable, and supposedly closed. However, it is not difficult to show that the set of points in P that are two-sided limit points of P must be dense in itself. In the present note we fill this gap by obtaining the following slightly more general result.

THEOREM 1. *Let A be a bounded, perfect, nowhere dense, liner set. There exist conformally equivalent slit domains Ω_1, Ω_2 whose boundaries $E_i = \partial\Omega_i$, $i = 1, 2$, have the following properties*:

(i) *The components of E_1 are not all points.*

(ii) *The projection of E_1 onto the x-axis is A.*

(iii) *E_2 is a compact, totally disconnected subset of the x-axis.*

To obtain an example for which (b) holds one then merely chooses

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A to have linear measure zero, for example, the Cantor middle-third set.

To construct E_1 and E_2 we shall first obtain some auxiliary results in §§2 and 3. The sets E_1 and E_2 are described in §4. Our construction follows the ideas of Koebe, the principal deviation from Koebe's work being the method of construction of the set here denoted by Σ.

Before proceeding with the proof of Theorem 1, it is of interest to note an immediate corollary. A significant property of the set E_2 of Theorem 1 can be stated in modern terminology if, following [1], we say that a compact set E in the z-plane is a null-set of class N_D if and only if, for any region Z of the z-plane containing E, every function $f(z)$, regular in $Z-E$, and possessing a finite Dirichlet integral there, can be extended to a function regular in Z. (The class N_D has been studied extensively by Sario [7], Ahlfors and Beurling [1], and others.) It is known that if E is a null-set of class N_D then the boundary of any region conformally equivalent to the complement of E with respect to the extended z-plane must be totally disconnected. Now E_1 contains a continuum, by (i). Hence, using the example $E=E_2$, with A arbitrary, we have the following.

Theorem 2. *There exists a linear, compact, totally disconnected set E which is not a null-set of class N_D.*

2. In what follows R denotes the real line. Given a set $A\subset R$, let $A_{II}=\{x \mid x\in A,\ x \text{ is the limit of points of } A \text{ both from the left and right}\}$. Let $A_I=A-A_{II}$.

Lemma 1. *If A is a perfect set in R, then A_{II} contains a perfect set.*

First Proof. Let $\{I_i\}$, $i=1, 2, \cdots$ be a (possibly terminating) enumeration of the components of $R-A$. Each set I_i is a finite or semi-infinite open interval. Note that $x\in A_I$ if and only if x is a one-sided limit point of A, that is, if and only if x is the finite endpoint of some interval I_i. Hence A_I is a finite or denumerable set. Since A is uncountable, $A_{II}=A-A_I$ is an uncountable Borel set. Therefore [2] A_{II} contains a perfect set.

Second Proof. Each finite endpoint of I_i is in A_I. On the other hand, each point of A is a condensation point. Therefore each finite endpoint of I_i is the limit of points of $A_{II}\cap(R-I_i)$. We shall use this remark in the course of defining a (possibly terminating) sequence $\{J_i\}$, $i=1, 2, \cdots$, of finite or semi-infinite open intervals, below. The definition is by induction.

(a) J_1 is an open interval whose finite endpoints are in A_{II}, and such that $J_1\supset \mathrm{Cl}[I_1]$. (The symbol Cl denotes closure.)

(b) Suppose $J_1, \cdots, J_n$ have been determined with the following properties:

(b1) $\mathrm{Cl}[J_i] \cap \mathrm{Cl}[J_j] = \emptyset$, unless $i=j$, $i, j=1, 2, \cdots, n$,

(b2) Each finite endpoint of J_k, $k=1, 2, \cdots, n$, is in A_{II}. (Consequently, by (b1), each $\mathrm{Cl}[I_i]$ is either in some J_k, $1 \leqq k \leqq n$, or in $R - \bigcup_1^n J_i$.)

If all $\mathrm{Cl}[I_i]$ are in $\bigcup_1^n J_i$ the process terminates. Otherwise, let $I_{i(n)}$ be the first interval in the enumeration $\{I_i\}$ whose closure lies in $R - \bigcup_1^n J_i$. In view of the remark at the beginning of the proof we can choose J_{n+1}, disjoint from $\bigcup_1^n \mathrm{Cl}[J_i]$, in such a way that each finite endpoint of J_{n+1} is in A_{II}, and $J_{n+1} \supset \mathrm{Cl}[I_{i(n)}]$.

The sequence $\{J_i\}$ resulting from the above process has the following properties.

(i) Each J_i is an open, finite or semi-infinite interval in R.

(ii) $\mathrm{Cl}[J_i] \cap \mathrm{Cl}[J_j] = \emptyset$, unless $i=j$.

(iii) Each set $\mathrm{Cl}[I_i]$ is contained in some J_j.

Conclusion (iii) is assured because the sequence $i(1), i(2), \cdots$ is strictly increasing. By (i) and (ii), $R - \bigcup J_i$ is a nonempty closed set. By (ii), this set has no isolated points, hence is perfect. By (iii), $R - \bigcup J_i \subset R - \bigcup \mathrm{Cl}[I_i] = A_{II}$, as required. For future reference we note that the sequence $\{J_i\}$ can be constructed in such a manner that

(iv) the number of semi-infinite intervals among $\{J_i\}$ that are semi-infinite to the left (right) does not exceed the number of semi-infinite intervals among $\{I_i\}$ that are semi-infinite to the left (right).

LEMMA 2. *If A and B are perfect sets in R, and $A_{II} \supset B$, then there exists a perfect set C such that $A_{II} \supset C \supset C_{II} \supset B$.*

PROOF. Let K be a component of $R - B$, and let $\{I_i(K)\}$ be an enumeration of the components of $K - A$. ($K - A$ may be empty, in which case the enumeration is also empty.) $A \cap K$ is a perfect set in the topology relative to K. Since K is homeomorphic with R, it follows from statements (i) to (iv) of the proof of Lemma 1 that there is a sequence $\{J_i(K)\}$ of subintervals of K such that, in the topology relative to K,

(i′) each $J_i(K)$ is an open subinterval of K whose finite endpoints fall in K,

(ii′) $\mathrm{Cl}[J_i(K)] \cap \mathrm{Cl}[J_j(K)] = \emptyset$, unless $i=j$,

(iii′) each $\mathrm{Cl}[I_i(K)]$ is contained in some $J_j(K)$.

Conclusion (i′) follows from (i) and (iv) of Lemma 1, because the finite endpoints of K lie in B, and hence also in A_{II}. This implies that no endpoint of any $I_i(K)$ coincides with a finite endpoint of K. Therefore, by (iv), translated to the present case by means of the homeomorphism, it follows that all $J_i(K)$ can be chosen so that (i′)

holds. In view of (i′), it actually follows that (i′) to (iii′) hold in the *topology of R*.

If $\{K_i\}$ is an enumeration of the components of $R-B$ then the set $C=R-\bigcup_{i,j}J_i(K_j)$ is perfect, by (i′) and (ii′). Moreover, $C_{II}=R-\bigcup_{i,j}\mathrm{Cl}[J_i(K_j)]\supset B$, since $\mathrm{Cl}[J_i(K_j)]\subset K_j$, by (i′). Also, $C\subset A_{II}$, by (iii′), as required.

LEMMA 3. *Suppose A is a perfect set in R, and H is a countable set, $H\subset[0, 1]$, $0\in H$, $1\in H$. There exists a sequence of perfect sets $\{A(y)\}$, $y\in H$, such that*

$$A(0) = A, \qquad A(z) \subset A(y)_{\mathrm{II}} \text{ whenever } y < z,\ y \in H,\ z \in H.$$

If A is totally disconnected, then $A(y)$ is totally disconnected for each $y\in H$.

PROOF. Suppose $H=\{y_i\}$, $i=1, 2, \cdots, 0=y_1, 1=y_2$. To construct $\{A(y)\}$ for $y\in H$ we start with $A(0)=A$. Lemma 1 guarantees the existence of a set $A(1)\subset A(0)_{II}$. Let z_i, w_i be the unique numbers among $y_1, y_2, \cdots, y_i$ closest to y_{i+1}, with $z_i<y_{i+1}<w_i$. To construct $A(y_{i+1})$, $i\geqq 2$, we apply Lemma 2, with $A(y_{i+1})$ corresponding to C, interpreting the A of Lemma 2 as $A(z_i)$, and the B of Lemma 2 as $A(w_i)$. Since $y\in H$ implies $A(y)\subset A$, it is clear that $A(y)$ is totally disconnected for each $y\in H$, if A is totally disconnected.

3. Let A be any perfect, totally disconnected set in $[0, 1]$. Using Lemma 3, we shall construct a *closed* set Σ lying in the unit square $\{x+iy\,|\,0\leqq x\leqq 1, 0\leqq y\leqq 1\}$ of the z-plane ($z=x+iy$) with the following properties.

(a) The components of Σ are closed linear segments (possibly points) L_ξ, parallel to the y-axis, with lower initial point on the x-axis, at $x=\xi$, $\xi\in A$.

(b) There is at least one component L_x (actually uncountably many) with length $|L_x|>0$.

(c) Any point in Σ, not at the top of a segment L_x, is the limit, both from the left and right, of points of Σ.

To construct Σ we identify the A of Lemma 3 with our present A, and choose as the H of Lemma 3 a countable set, such as the rationals in $[0, 1]$, whose closure is $[0, 1]$. The sets $A(y)$ of Lemma 3 will be employed essentially as cross-sections of our set Σ. Namely, we define

$$\Sigma = \bigcup_{\xi\in A} L_\xi,$$

where L_ξ is a closed vertical segment (possibly a point) with lower initial point on the x-axis, at $x=\xi$, and

$$|L_x| = \sup\{y \mid x \in A(y), y \in H\}.$$

To show that Σ is closed, suppose $z_0 = x_0 + iy_0 \in \text{Cl}[\Sigma]$. There then exist points $x_n + iy_n \in \Sigma$, $n = 1, 2, \cdots$, such that $x_0 = \lim x_n$, $y_0 = \lim y_n$. Since $x_n + iy_n \in \Sigma$ implies $x_n \in A$, it follows that $x_0 \in A$, because A is closed. The case $y_0 = 0$ is immediately disposed of by noting that, since $x_0 \in A$, $z_0 = x_0 + i0 \in \Sigma$. If $y_0 > 0$, let $\delta > 0$ be arbitrary. By the density property of H, there exist δ', δ'', $0 < \delta' < \delta'' < \delta$, such that $y_0 - \delta'' \in H$. Now $y_n > y_0 - \delta'$ for $n > N(\delta')$. Therefore, from the definition of Σ, $x_n + i(y_0 - \delta') \in \Sigma$ for $n > N(\delta')$. Therefore, $x_n \in A(y_0 - \delta'')$ for $n > N(\delta')$. Since $A(y_0 - \delta'')$ is closed, $x_0 \in A(y_0 - \delta'')$. Thus $|L_{x_0}| \geqq y_0 - \delta'' > y_0 - \delta$. Hence $|L_{x_0}| \geqq y_0$. Therefore, $x_0 + iy_0 \in \Sigma$.

A point is in Σ if and only if it lies on some segment L_ξ. Hence, if a component K of Σ contained $L_\xi \cup L_\eta$, $\xi < \eta$, then K would have to contain $\{x + iy \mid \xi \leqq x \leqq \eta, y = 0\}$. But this is impossible, because A is totally disconnected. Hence we have property (a).

Property (b) follows from the fact that $A(1)$ is perfect, hence contains at least one point (actually uncountably many), say $\lambda \in A(1)$. Thus, $|L_\lambda| = 1$, $L_\lambda \subset \Sigma$.

To prove (c), assume $x_0 + iy_0 \in \Sigma$, $y_0 < |L_{x_0}|$. Let us choose $\eta_1, \eta_2 \in H$, $y_0 < \eta_1 < \eta_2 < |L_{x_0}|$. We have $x_0 \in A(\eta_2)$. Therefore, $x_0 \in A(\eta_1)_{II}$. Therefore, there exist sequences $\{\xi_n\}$, $\{x_n\}$, of points of $A(\eta_1)$ such that $\xi_n < x_0 < x_n$, $\lim \xi_n = \lim x_n = x_0$. Property (c) follows from the fact that $\xi_n + iy_0$ and $x_n + iy_0$ lie in Σ.

4. Let Ω be the region

$$\{z \mid \mathcal{I}z > 0\} - \Sigma, \qquad (z = x + iy),$$

where Σ is the set found in §3. By the Riemann mapping theorem there exists a schlicht function, $w = f(z)$, $f(\infty) = \infty$, mapping Ω onto the upper half w-plane. Since the limiting values of f on the linear open set $R - A$ of the x-axis are real, it follows, by the strong form of the Schwarz reflection principle, that there is an extension of f to a function schlicht in

$$\Omega_1 = \Omega \cup \Omega^* \cup \{z \mid \mathcal{I}z = 0, \mathcal{R}z \in R - A\} \cup \{\infty\},$$

with $f(\infty) = \infty$, $f(z^*) = f(z)^*$. (* denotes reflection in the real axis.) Evidently, $E_1 = \partial\Omega_1 = \Sigma \cup \Sigma^*$. The set of limiting values, E_2, of f on E_1 is real, and according to Carathéodory's prime end theory [3], if $\Omega_2 = f(\Omega_1)$, then $E_2 = \partial\Omega_2$. In view of the reflection principle, E_2 is also the set of limiting values on Σ of the restriction of f to Ω. By property (c), §3, each segment L_x, $x \in A$, with $|L_x| > 0$, is the impression of a prime end (of the second type) of $\partial\Omega$. The remaining points of Σ (all on the x-axis) are accessible points of $\partial\Omega$. Furthermore, since A is totally disconnected, there are accessible points of $\partial\Omega$ in $R-A$ between any two prime ends of Σ. Hence,by Carathéodory's theory, E_2 is a totally disconnected set. This completes the proof of Theorem 1.

The fact that E_2 is totally disconnected makes it possible to interpret E_2 not only as the boundary of a (parallel) slit domain, but also, for instance, as the boundary of a circular slit domain. For example, by deleting E_2 along a radius from an annulus (or disk) one obtains an example of a circular slit annulus (or disk) which can be shown to be not minimal in the sense of [4] or [6].

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