ON THE COMPACTNESS OF THE STRUCTURE SPACE OF A RING

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If A is a ring, we denote by $\mathcal{S}=\mathcal{S}(A)$ the set of all primitive ideals of A. A topology may be defined on $\mathcal{S}$ in the following way: For $\mathfrak{J}\subseteq\mathcal{S}$, the closure of $\mathfrak{J}$ in $\mathcal{S}$ is the set of all $P\in\mathcal{S}$ such that $P\supseteq\bigcap\mathfrak{J}$. With this topology, $\mathcal{S}$ is the *structure space* [4] of A. It is well known that if A has a unit element, then $\mathcal{S}$ is compact [4; 5, p. 208]. Moreover, M. Schreiber [7] has recently observed that if every (two-sided) ideal of A is finitely generated, then $\mathcal{S}$ is again compact. However, since the condition that A have a unit element neither implies nor is implied by the condition that every ideal of A be finitely generated, it is clear that neither of these conditions is necessary in order that $\mathcal{S}$ be compact. In Theorem 2 of the present note we present a condition that is both necessary and sufficient for the compactness of $\mathcal{S}$. The sufficient conditions already cited are both immediate consequences of our result. Although a direct proof of Theorem 2 can easily be formulated, we have chosen to obtain Theorem 2 as a consequence of a general lattice-theoretic result which we state as Theorem 1.[1]

1. A lattice-theoretic result. The appropriate setting for the present discussion is a family $\mathcal{L}$ of sets that forms a complete lattice relative to set inclusion. The lattice operations of join and meet in $\mathcal{L}$ will be denoted by $\vee$ and $\wedge$; the symbols $\cup$ and $\cap$ will be reserved for set-theoretic union and intersection.

If $\mathcal{S}$ is a fixed subset of $\mathcal{L}$ and if $\mathfrak{J}\subseteq\mathcal{S}$, then the *closure* $\mathfrak{J}^-$ of $\mathfrak{J}$ in $\mathcal{S}$ is the set of all $P\in\mathcal{S}$ such that $P\supseteq\bigwedge\mathfrak{J}$. If $\mathfrak{J}\subseteq\mathcal{S}$, then $\mathfrak{J}$ is *closed* in case $\mathfrak{J}=\mathfrak{J}^-$, and $\mathcal{S}$ is *compact* in case each family of closed subsets of $\mathcal{S}$ having the finite intersection property has a nonempty intersection (cf. [6, p. 780]).

REMARK. The mapping $\mathfrak{J}\to\mathfrak{J}^-$ is, in any event, a closure operation [1, p. 49] on the subsets of $\mathcal{S}$. The case of greatest interest, however, is that in which each $P\in\mathcal{S}$ is *strongly irreducible* [2] in the sense that (i) P is not the unit of $\mathcal{L}$ and (ii) for every B and C in $\mathcal{L}$, $B\wedge C\subseteq P$ only if either $B\subseteq P$ or $C\subseteq P$. In this case the mapping $\mathfrak{J}\to\mathfrak{J}^-$ is a Kuratowski closure operation and therefore defines a topology (the

Received by the editors December 2, 1958.

[1] As observed in [2], topologies of the sort described above can be treated most naturally in a lattice-theoretic setting. For results related to Theorem 1, see §3 of J. G. Horne, Jr., *On the ideal structure of certain semirings and compactification of topological spaces*, Trans. Amer. Math. Soc. vol. 90 (1959) pp. 408–430.

Stone topology) on $\mathcal{S}$ (see [2, Theorem 1.1]). (Conversely, if $\mathfrak{I}\to\mathfrak{I}^-$ defines a topology on $\mathcal{S}$, then it is easy to see that each $P\in\mathcal{S}$ must be strongly irreducible.) As an example, let $\mathcal{L}$ be the lattice of all ideals of a ring A; then each $P\in\mathcal{S}(A)$ is prime [5, p. 195] and therefore strongly irreducible in $\mathcal{L}$.

The following theorem generalizes Theorem 3.4 of [2].

THEOREM 1. *Let $\mathcal{L}$ be a family of sets that forms a complete lattice relative to set inclusion, let $\mathcal{S}$ be a fixed subset of $\mathcal{L}$, and consider the following three conditions*:

(I) *If $B_\alpha\in\mathcal{L}$ for each α in some index set Ω and if $x\in\bigvee\{B_\alpha:\alpha\in\Omega\}$, then $x\in\bigvee\{B_\alpha:\alpha\in\Phi\}$ for some finite subset Φ of Ω.*

(II) *If $x\in\bigcup\mathcal{L}$, then $x\in\bigwedge\{P\in\mathcal{S}: x\in P\}$.*

(III) *For each $B\in\mathcal{L}$ such that $B\not\subseteq P$ for every $P\in\mathcal{S}$ there exists a finite subset F of B such that $F\not\subseteq P$ for every $P\in\mathcal{S}$.*

If $\mathcal{L}$ satisfies (I) *and* (III), *then $\mathcal{S}$ is compact. Conversely, if $\mathcal{L}$ satisfies* (II) *and if $\mathcal{S}$ is compact, then $\mathcal{L}$ satisfies* (III).

PROOF. Suppose first that $\mathcal{L}$ satisfies (I) and (III) and let $\{\mathfrak{F}_\alpha:\alpha\in\Omega\}$ be a family of closed subsets of $\mathcal{S}$ with empty intersection. Then $\bigvee\{\bigwedge\mathfrak{F}_\alpha:\alpha\in\Omega\}\not\subseteq P$ for every $P\in\mathcal{S}$, and this property is inherited by some finite subset $F=\{x_1,\cdots,x_n\}$ of $\bigvee\{\bigwedge\mathfrak{F}_\alpha:\alpha\in\Omega\}$. Now for each $i=1,\cdots,n$ we have $x_i\in\bigvee\{\bigwedge\mathfrak{F}_\alpha:\alpha\in\Phi_i\}$ for some finite subset Φ_i of Ω. Setting $\Phi=\bigcup_{i=1}^n\Phi_i$, we then have $F\subseteq\bigvee\{\bigwedge\mathfrak{F}_\alpha:\alpha\in\Phi\}$, from which it follows that $\bigcap\{\mathfrak{F}_\alpha:\alpha\in\Phi\}=\varnothing$. Therefore $\mathcal{S}$ is compact.

Conversely, suppose that $\mathcal{S}$ is compact and that $\mathcal{L}$ satisfies (II). Let B be an element of $\mathcal{L}$ with the property that $B\not\subseteq P$ for every $P\in\mathcal{S}$. From (II) it follows that for each $x\in\bigcup\mathcal{L}$ the set $\mathfrak{F}_x=\{P\in\mathcal{S}: x\in P\}$ is closed in $\mathcal{S}$. Moreover, it is clear that $\bigcap\{\mathfrak{F}_x: x\in B\}=\varnothing$ so that, by the compactness of $\mathcal{S}$, $\bigcap\{\mathfrak{F}_x: x\in F\}=\varnothing$ for some finite subset F of B. From this it is immediate that $F\not\subseteq P$ for every $P\in\mathcal{S}$, and we conclude that $\mathcal{L}$ satisfies (III). The proof is now complete.

REMARKS. (1) Condition (I) above is equivalent to the following condition (I′): $\bigvee\mathcal{C}=\bigcup\mathcal{C}$ *for every chain* $\mathcal{C}$ *in* $\mathcal{L}$. (The implication (I) →(I′) is readily verified; the implication (I′)→(I) is a consequence of Lemma 3.2 of [2].[2]) We note also that condition (II) is satisfied if we assume that $\bigcap\mathfrak{I}\in\mathcal{L}$ for every $\mathfrak{I}\subseteq\mathcal{S}$. In particular, then, both (I) and (II) are satisfied (for any choice of $\mathcal{S}$) by the lattice of all ideals of a ring, or, more generally, by the lattice of all subalgebras

[2] The hypothesis of Lemma 3.2 of [2] includes the assumption that $B\wedge C=B\cap C$ for all B, $C\in\mathcal{L}$; this assumption, however, is not used in the proof.

of an abstract algebra with finitary operations [1, p. vii].

(2) If $\mathcal{S}\subseteq\mathcal{L}$ and if $\mathfrak{m}$ is any infinite cardinal, then $\mathcal{S}$ is $(\mathfrak{m}, \infty)$-*compact* (cf. [3]) in case for each family $\{\mathfrak{F}_\alpha: \alpha\in\Omega\}$ of closed subsets of $\mathcal{S}$ having empty intersection there is a subset Φ of Ω with card $\Phi\leqq\mathfrak{m}$ and $\bigcap\{\mathfrak{F}_\alpha: \alpha\in\Phi\}=\varnothing$. (For example, if $\mathcal{S}$ is an $(\aleph_0, \infty)$-compact set of strongly irreducible elements, then $\mathcal{S}$ is a Lindelöf space.) We remark that Theorem 1 still holds if we replace, in its statement, "compact" by "$(\mathfrak{m}, \infty)$-compact" and "finite subset F" by "subset F of cardinality $\leqq\mathfrak{m}$." The proof requires only trivial modifications; in particular, in its first paragraph we must use the fact that if a set Φ is a union of $\mathfrak{m}$ or fewer finite sets, then card $\Phi\leqq\mathfrak{m}$.

(3) Neither of the two hypotheses, in either of the two assertions of the theorem, can be omitted. This is shown by the following three examples:

EXAMPLE 1. Let A be a ring whose structure space $\mathcal{S}$ is not compact and let $\mathcal{L}$ be the lattice of ideals of A. Then $\mathcal{L}$ satisfies (I) and (II) but not (III).

EXAMPLE 2. Let X be an infinite compact Hausdorff space, let $\mathcal{L}$ be the lattice of all open subsets of X, and let $\mathcal{S}$ be the set of all strongly irreducible elements of $\mathcal{L}$. Then $\mathcal{S}$ is precisely the set of all complements of points of X and the mapping $x\rightarrow X-\{x\}$ is a homeomorphism of X onto $\mathcal{S}$ [2, Theorem 2.8]. Then $\mathcal{S}$ is compact, and, as is readily seen, $\mathcal{L}$ satisfies (I) but neither (II) nor (III).

EXAMPLE 3. Let I be the closed unit interval $[0, 1]$, let $C(I)$ be the ring of all real-valued continuous functions on I, and, for $x\in I$, denote by M_x the maximal ideal of $C(I)$ consisting of all $f\in C(I)$ such that $f(x)=0$. Let $\mathcal{L}$ be the lattice consisting of $C(I)$ together with all intersections of maximal ideals of $C(I)$ (an alternative description: $\mathcal{L}$ is the lattice of all uniformly closed ideals of $C(I)$), and let $\mathcal{S}=\{M_x: 0\leqq x<1\}$. As is well known, the mapping $x\rightarrow M_x$ is a homeomorphism of I onto the space of all maximal ideals of $C(I)$. It follows that $\mathcal{S}$ is not compact and that $\mathcal{L}$ satisfies (II) and (III) but not (I).

2. **Applications to rings.** If B is an ideal in a ring A, then $B\not\subseteq P$ for every $P\in\mathcal{S}(A)$ if and only if A/B is a radical ring (see e.g. [5, p. 205]). In view of Remark (1) above, we therefore have the following special case of Theorem 1:

THEOREM 2. *Let A be a ring and let $\mathcal{S}$ be the structure space of A. Then $\mathcal{S}$ is compact if and only if for each ideal B in A such that A/B is a radical ring, B contains a finitely generated ideal I such that A/I is a radical ring.*

It is clear from Theorem 2 that if every ideal of A is finitely generated, then $\mathcal{S}$ is compact [7, Theorem 2]. On the other hand, if $\mathcal{S}$ is compact, then A/I is a radical ring for some finitely generated ideal I, from which it follows [5, p. 206] that $\mathcal{S}(A)$ is homeomorphic to $\mathcal{S}(I)$; this is Theorem 3 of [7].

An important class of rings consists of those rings A such that no nonzero homomorphic image of A is a radical ring. For such a ring, the condition set forth in Theorem 2 is equivalent to the condition that A itself be generated, as an ideal, by a finite number of elements. We therefore have the following corollary:

COROLLARY. *Let A be a ring with the property that no nonzero homomorphic image of A is a radical ring and let $\mathcal{S}$ be the structure space of A. Then $\mathcal{S}$ is compact if and only if A is generated, as an ideal, by a finite number of elements.*

If A has a unit element, then no nonzero homomorphic image of A is a radical ring and we conclude that $\mathcal{S}$ is compact.

We remark that there exist rings A with the property that $\mathcal{S}(A)$ is compact but such that A does possess a nonzero homomorphic image that is a radical ring. To see this, let B be a semi-simple ring such that $\mathcal{S}(B)$ is compact, let R be a (nonzero) radical ring, and let A be the direct sum $B \oplus R$. Since R is the radical of A, it follows [5, p. 205] that $\mathcal{S}(A)$ is homeomorphic to $\mathcal{S}(A/R)$, and the latter is in turn homeomorphic to $\mathcal{S}(B)$.

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