ON THE PROXIMAL RELATION IN TOPOLOGICAL DYNAMICS

JOSEPH AUSLANDER[1]

Let (X, T) be a transformation group with compact Hausdorff phase space X. The points x and y of X are said to be *proximal* provided, whenever β is a member of the unique compatible uniformity of X, there exists $t \in T$ such that $(xt, yt) \in \beta$. If x and y are not proximal, they are said to be *distal.*

Let P denote the proximal relation in X. P is a reflexive, symmetric, T invariant relation, but is not in general transitive or closed. As is customary, if $x \in X$, let $P(x) = [y \in X \mid (x, y) \in P]$.

If $x \in X$, the *orbit of* x is the set $xT = [xt \mid t \in T]$. The closure of xT, denoted by $(xT)^-$ is called *orbit closure* of x. A nonempty subset M of X is said to be a *minimal orbit closure*, or *minimal set*, if $M = (xT)^-$ for all $x \in M$. If A is a nonempty closed, T invariant subset of X, then A contains at least one minimal set [**3**, 2.22].

We may consider T as a subset of X^X. (We identify two elements t_1 and t_2 of T if $xt_1 = xt_2$ for all $x \in X$.) Let E be the closure of T in X^X. E is a compact semigroup (but not a topological semigroup); it is called the *enveloping semigroup of* (X, T).

The enveloping semigroup of a transformation group was defined in [**2**]. Its algebraic properties, and their connection with the recursive properties of the transformation group are studied in [**1**].

A nonempty subset I of E is called a *right ideal* in E if $IE \subset I$. If I contains no proper nonempty subsets which are also right ideals, I is called a *minimal right ideal.*

In Lemma 1, we summarize some results from [**1**] which we shall repeatedly use in this paper.

LEMMA 1. (i) *If K is a closed right ideal in E, K contains a minimal right ideal I.*

(ii) *If I is a minimal right ideal in E, then I is closed.*

(iii) *If I is a minimal right ideal in E, and $x \in X$, then xI is a minimal set in X.*

(iv) *If K is a nonempty closed set in E such that $K^2 \subset K$, then K contains an idempotent (i.e., an element u such that $u^2 = u$).*

Presented to the Society, January 29, 1960; received by the editors December 22, 1959.

[1] This work was supported in part by a National Science Foundation Research Grant.

(v) *If I is a minimal right ideal in E, u an idempotent in I, and $p \in I$, then $up = p$.*

(vi) *The points x and y of X are proximal if and only if there exists a minimal right ideal I in E such that $xp = yp$ for all p in I.*

(vii) *P is an equivalence relation in X if and only if E contains exactly one minimal right ideal.*

Observe that (i) tells us that E always contains at least one minimal right ideal.

LEMMA 2. *Let $x \in X$, and let M be a minimal set contained in $(xT)^-$. Then there exists a minimal right ideal I in E such that $M = xI$. Moreover, there is a point $y \in M$ such that x and y are proximal.*

PROOF. Let $F = [p \in E \mid xp \in M]$. If $p \in F$ and $q \in E$, then $xpq \in Mq \subset M$, so $pq \in F$. Therefore, $FE \subset F$. Also, since M is closed in X, F is closed in E. That is, F is a closed right ideal in E. By Lemma 1 (i), F contains a minimal right ideal I. Then $xI \subset xF \subset M$. By Lemma 1 (iii), xI is a minimal set in X. Consequently, $xI = M$.

Now, let u be an idempotent in I. Then $y = xu \in xI = M$, and $yu = xu^2 = xu$, so x and y are proximal.

THEOREM 1. *Suppose that P is an equivalence relation in X. Then*

(i) *$X = \bigcup_\alpha N_\alpha$, where the N_α are pairwise disjoint, $N_\alpha T \subset N_\alpha$, and each N_α contains precisely one minimal set M_α.*

(ii) *If $x \in N_\alpha$, then x is proximal to a point $y \in M_\alpha$.*

(iii) *If P is closed in $X \times X$, then the sets N_α are closed.*

PROOF. (i) Let $x \in X$. Since proximal is an equivalence relation, E contains just one minimal right ideal by Lemma 1, (vii). Therefore, by Lemma 2, $(xT)^-$ contains just one minimal set.

Let $\{M_\alpha\}$ be the class of minimal sets in X. Let

$$N_\alpha = [x \in X \mid (xT)^- \supset M_\alpha].$$

It now follows that the N_α are pairwise disjoint, and that their union is X. By definition, each N_α contains just one minimal set, namely M_α. It is clear that $N_\alpha T \subset N_\alpha$.

(ii) This is an immediate consequence of Lemma 2, since $M_\alpha \subset (xT)^-$ whenever $x \in N_\alpha$.

(iii) Suppose that P is closed. Let $\{x_n \mid n \in D\}$ be a net in N_α, and suppose $x_n \to x$. Then there exists $y_n \in M_\alpha$ such that $(x_n, y_n) \in P$. By choosing an appropriate subnet if necessary, let $y_n \to y \in M_\alpha$. Since P is closed $(x, y) \in P$. Therefore $M_\alpha = (yT)^- \subset (xT)^-$ and $x \in N_\alpha$. Hence N_α is closed.

The following example shows that if P is not closed the sets N_α need not be closed. The space X is a compact subset of the plane. X is the union of the circles C_n, $(n=0, 1, 2, \cdots)$ defined by $x^2+(y-1+n/(n^2+1))^2=(1-n/(n^2+1))^2$. A point Q on C_n will be given the coordinates (α, n) where α is the angle between 0 and π formed by the positive x axis and the chord from the origin to Q.

We define a homeomorphism f of X onto X by $(\alpha, n)f=(\alpha+(1/n)\sin\alpha, n)$, if $n>0$. We define $(\alpha, 0)f=(\alpha, 0)$. The group T consists of the positive and negative powers of f. It is easily shown that P is an equivalence relation in X.

If $0\leqq\alpha<\pi$, let M_α be the set consisting of the point $(\alpha, 0)$. Then the sets M_α are the minimal sets of X. If $0<\alpha<\pi$, $N_\alpha=M_\alpha$. However, $N_0=(0, 0)\cup C_1\cup C_2\cup\cdots$, which is not closed.

Let u and v be idempotents in E. We write $u\sim v$ if $uv=u$ and $vu=v$. It is easy to see that $\sim$ is an equivalence relation. In [1] it is proved that if I_1 and I_2 are minimal right ideals in E, and u is an idempotent in I_1, then there is precisely one idempotent u_2 in I_2 such that $u_1\sim u_2$.

Let (X, T) and (Y, T) be transformation groups with the same phase group T. A *homomorphism* of (X, T) onto (Y, T) is a continuous map ϕ of X onto Y such that $(xt)\phi=(x\phi)t$ for all $x\in X$ and all $t\in T$.

Let $E(X)$ and $E(Y)$ be the enveloping semigroups of the transformation groups (X, T) and (Y, T) respectively. It is proved in [2] that if ϕ is a homomorphism of (X, T) onto (Y, T), then the map $\theta: E(X)\to E(Y)$ defined by $(x\phi)(p\theta)=(xp)\phi(x\in X, p\in E(X))$ is a continuous onto semigroup homomorphism.

Let $((X_i, T)\mid i\in\mathscr{I})$ be a family of transformation groups, with the same phase group T. Let $X=\times_{i\in\mathscr{I}} X_i$, and let $x=(x_i\mid i\in\mathscr{I})\in X$. If $t\in T$ define $xt=(x_it\mid i\in\mathscr{I})$. With this definition of xt, (X, T) is a transformation group.

THEOREM 2. *Let $((X_i, T)\ i\in\mathscr{I})$ be a family of transformation groups with compact Hausdorff phase space, and the same phase group T. Let $X=\times_{i\in\mathscr{I}} X_i$. Then proximal is an equivalence relation in X if and only if proximal is an equivalence relation in each X_i.*

PROOF. Suppose proximal is an equivalence relation in each X_i. Let π_i be the projection of X onto X_i. Then π_i is a homomorphism of (X, T) onto (X_i, T). Let θ_i be the induced homomorphism of $E(X)$ onto $E(X_i)$.

Now suppose that I and I' are distinct minimal right ideals in $E(X)$. Then there are idempotents $u\in I$ and $u'\in I'$ with $u\sim u'$. Then $u\theta_i$ and $u'\theta_i$ are idempotents in $I\theta_i$ and $I'\theta_i$ respectively such that $u\theta_i\sim u'\theta_i$. Since proximal is an equivalence relation in X_i, $I\theta_i=I'\theta_i$

and therefore $u\theta_i = u'\theta_i$. Then, if $x \in X$, $(xu)\pi_i = (x\pi_i)(u\theta_i) = (x\pi_i)(u'\theta_i) = (xu')\pi_i$. This is true for every $i \in \mathcal{I}$. That is $xu = xu'$, and since x is arbitrary $u = u'$. But then $I \cap I' \neq \emptyset$. Therefore, $E(X)$ has only one minimal right ideal, and proximal is an equivalence relation in X.

Now suppose proximal is an equivalence relation in X. Let $j \in \mathcal{I}$, and suppose proximal is not an equivalence relation in X_j. Let I_j and I_j' be distinct minimal right ideals in $E(X_j)$. Since I_j and I_j' are closed and disjoint, $I_j\theta_j^{-1}$ and $I_j'\theta_j^{-1}$ are closed disjoint right ideals in $E(X)$. Hence they contain minimal right ideals I and I'. This is a contradiction.

Let $\mathcal{P}$ be a property of a transformation group such that

(1) $\mathcal{P}$ is hereditary. (That is, if (Y, T) has property $\mathcal{P}$, and M is an invariant subset of Y, then (M, T) has property $\mathcal{P}$.)

(2) $\mathcal{P}$ is productive. (That is, if $((X_i, T) \mid i \in \mathcal{I})$ is a family of transformation groups, each of which has property $\mathcal{P}$, then $(\times_{i \in \mathcal{I}} X_i, T)$ has property $\mathcal{P}$.)

Then, if (X, T) is *any* transformation group, there exists a smallest closed, T invariant equivalence relation R in X such that the transformation group $(X/R, T)$ has property $\mathcal{P}$ [**2**, Remark 8].

Let $\mathcal{P}$ denote the property "proximal is an equivalence relation." Obviously $\mathcal{P}$ is hereditary, and Theorem 2 tells us that $\mathcal{P}$ is productive. Hence, we may always divide out appropriately so that in the quotient transformation group proximal is an equivalence relation.

If $p \in E$ and $z = (x, y) \in X \times X$, we define $zp = (xp, yp)$. With this definition, we may consider p as a member of the enveloping semigroup of $X \times X$. If $R \subset X \times X$ and $H \subset E$ then RH is the set $[zp \mid z \in R, p \in H]$, where zp is defined as above.

The diagonal of $X \times X$, that is, the set $[(x, x) \mid x \in X]$ will be denoted by Δ.

Observe that if $z = (x, y) \in X \times X$, then $z \in P$ if and only if $(zT)^- \cap \Delta \neq \emptyset$.

THEOREM 3. *The following are equivalent.*

(i) *P is an equivalence relation in X.*

(ii) *Every orbit closure in $(X \times X, T)$ contains precisely one minimal set.*[2]

(iii) *$PE \subset P$.*

(iv) *$PL \subset P$, where $L = [\cup I \mid I$ a minimal right ideal in $E]$.*

(v) *$PL \subset \Delta$.*

PROOF. We show that each statement in the theorem implies the following one and that (v) implies (i).

[2] This condition was suggested to me by Professor Robert Ellis.

PROOF. If (i) holds, then by Theorem 2, proximal is an equivalence relation in $X\times X$. By Theorem 1 (i) every orbit closure in $X\times X$ contains precisely one minimal set.

Suppose (ii) holds. Let $z=(x, y)\in P$, and let $q\in E$. We show that $zq\in P$. By the remark above, it is sufficient to show that $(zqT)^-\cap\Delta \neq\emptyset$. Let M be a minimal set contained in $(zqT)^-$. Since $zq\in(zT)^-$, $(zqT)^-\subset(zT)^-$ and therefore $M\subset(zT)^-$. Now $z\in P$, so $(zT)^-\cap\Delta\neq\emptyset$. Since $(zT)^-\cap\Delta$ is a nonempty closed T invariant set, it contains a minimal set M'. By (ii) $(zT)^-$ contains just one minimal set, so $M=M'$. Therefore, $M'\subset(zqT)^-$ and $(zqT)^-\cap\Delta\neq\emptyset$.

That (iii) implies (iv) is obvious.

Suppose (iv) is true. Let $z=(x, y)\in P$ and let $q\in L$. Then q is in some minimal right ideal I. By (iv), $zq=(xq, yq)\in P$. That is, xq is proximal to yq. By Lemma 1 (vi), there is a minimal right ideal I' in E such that $xqr=yqr$, for all $r\in I'$. Now qI' is a right ideal, and $qI'\subset IE\subset I$, so we must have $qI'=I$. Hence there is an $r\in I'$ such that $qr=q$. Therefore, $xq=yq$, and $zq=(xq, yq)\in\Delta$. Therefore, $PL\subset\Delta$.

Finally, suppose (v) holds. Let $(x, y)\in P$ and $(y, z)\in P$. It follows from (v) that $xq=zq$ for all $q\in L$, so $(x, z)\in P$.

COROLLARY 1. *If P is closed in $X\times X$, then P is an equivalence relation.*

PROOF. Let $z=(x, y)\in P$, and let $q\in E$. Let $\{t_n | n\in D\}$ be a net in T such that $t_n\to q$. Now $(xt_n, yt_n)\in P$, and since P is closed, $zq=(xq, yq)\in P$. Therefore, (iii) of Theorem 3 is satisfied.

The example given above shows that the converse of Corollary 1 is not true. For example, the points $(\pi/2, n)$ and $(0, 0)$ $(n=1, 2, \cdots)$, are proximal. But $(\pi/2, n)\to(\pi/2, 0)$ as $n\to\infty$ and $(\pi/2, 0)$ is clearly not proximal to $(0, 0)$ since both of these points are mapped into themselves by f.

If X is minimal under T, it is not known if P must be closed when P is an equivalence relation.

If I is a minimal right ideal in E, let $J(I)$ denote the set of idempotents in I. Let

$$J = \bigcup\, [J(I) \mid I \text{ a minimal right ideal}].$$

The transformation group (X, T) is said to be *pointwise almost periodic* if, for every $x\in X$, $(xT)^-$ is a minimal set.

THEOREM 4. *Suppose (X, T) is pointwise almost periodic, and let $x\in X$. Then*

(i) $P(x)=xJ$.

(ii) *Let I be a minimal right ideal in E. The points $xJ(I)$ are mutually proximal. If y is proximal to all $x' \in xJ(I)$, then $y \in xJ(I)$. (That is, the sets $xJ(I)$ are "maximal" sets of mutually proximal points.)*

(iii) *Let I be a minimal right ideal in E. Let $q \in I$ such that $(x, xq) \in P$ for all $x \in X$. Then $q \in J(I)$.*

PROOF. (i) It is clear that $xJ \subset P(x)$. Now, suppose $y \in P(x)$. Then there is a minimal right ideal I in E such that $xr = yr$, for all $r \in I$. Let $F = [r \in I \mid yr = y]$. Since $(yT)^-$ is minimal, $(yT)^- = yI$, and $F \neq \emptyset$. Now F is closed, and $F^2 \subset F$. Therefore, F contains an idempotent u. That is, there is a $u \in J(I)$ such that $yu = y$. But $xu = yu = y$, so $y \in xJ(I)$.

(ii) Suppose $u, v \in J(I)$. By Lemma 1 (v), $uv = v$. Therefore, $xuv = xv = xv^2 = xvv$, so $(xu, xv) \in P$.

If $(y, x') \in P$, for all $x' \in xJ(I)$, then in particular $(y, x) \in P$. Therefore, by (i), there exists a minimal right ideal I' in E and a $u' \in J(I')$ such that $y = xu'$. Let $u \in J(I)$ such that $u \sim u'$. By hypothesis, $(xu', xu) = (y, xu) \in P$. Then, for some minimal right ideal I'' in E, $xu'p'' = xup''$ for all $p'' \in I''$. Let $u'' \in J(I'')$ such that $u \sim u' \sim u''$. Then $xu'u'' = xuu''$. But $u'u'' = u'$, and $uu'' = u$, so $y = xu' = xu \in xJ(I)$.

(iii) Note that for all $x \in X$, and all $u \in J(I)$, $(xu, xq) \in P$. For by hypothesis, $(xu, xuq) \in P$. By Lemma 1 (v), $uq = q$, so $xuq = xq$.

Now, fix $x \in X$ and let $y = xq$. Then y is proximal to xu, for all $u \in J(I)$. Hence, by (ii) $xq = y = xv$, where $v \in J(I)$ and v depends (apparently) on x.

Now, by [1, Lemma 2 (3)] there is a $w \in J(I)$ such that $qw = q$. Since $vw = w$, we have $xq = xqw = xvw = xw$. That is, for all $x \in X$, $xq = xw$, so $q = w \in J(I)$.

REFERENCES

1. Robert Ellis, *A semigroup associated with a transformation group*, Trans. Amer. Math. Soc. vol. 94 (1960) pp. 272–281.

2. Robert Ellis and W. H. Gottschalk, *Homomorphisms of transformation groups*, Trans. Amer. Math. Soc. vol. 94 (1960) pp. 258–271.

3. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 36, 1955.

CARNEGIE INSTITUTE OF TECHNOLOGY