ON A CONJECTURE OF KOCH[1]

R. P. HUNTER

Let X be a topological space. We recall that D, a subset of X is called a C-set if any continuum which meets D and its complement must contain D.

Let S be a continuum which is a topological semigroup with identity 1, and let H denote the maximal subgroup of S containing 1. It is well known that H exists and is compact. The following four conjectures have been raised and shown to be equivalent by Koch, [2].[2]

(1) The unit is not a weak-cutpoint.

(2) S is aposyndetic at any point with respect to 1.

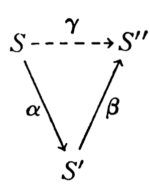
(3) The identity component of H is not a nontrivial C-set.

(4) 1 belongs to no nontrivial C-set.

We give here an affirmative answer to these conjectures. (We assume, of course, that S is not a group.)

THEOREM. *Let G be a compact invariant subgroup of H such that H/G is a Lie group. Then S contains a continuum M such that M meets H and the complement of H and such that $M \cap H \subseteq G$.*

PROOF. We consider H as a transformation group of S in the obvious way. Letting $H' = H/G$ and letting S' denote the space of orbits of G, we may consider H' as a transformation group of S'. Finally letting S'' denote the space of orbits under H itself, we have the following diagram.



,

where $\gamma = \alpha\beta$, and α, β, and γ, are all canonical mappings, as S'' may also be considered as the space of orbits of S' under H'. Since the decompositions defined by the sets $\{xH\}$ or by $\{xG\}$, $x \in S$, are con-

Received by the editors March 2, 1960.

[1] The author holds an S. H. Moss Postdoctoral Fellowship.

[2] Koch in [2] had affirmed this conjecture in case S was either homogeneous or one dimensional.

tinuous, the mappings γ and α are both open. It follows that β is open also.

Now H' is a compact Lie group of transformations acting on a compact connected space S'. Gleason [1, Theorem 3.3] has shown that there is a closed neighborhood N of $1'$ such that the orbit of any point of N meets a certain set L in precisely one point. That is to say there is a closed neighborhood N and a closed set L such that $nH' \cap L$ is a single point for each $n \in N$, i.e. a local cross section at $1'$. ($1'$, of course, denotes the identity of S'.)

Now let $\Delta = \beta|_L$. It is easy to see that Δ is a homeomorphism between L and $\beta(N)$. Letting N^0 denote the interior of N, we note that $\beta(N^0)$ is an open set about the point $\beta(1') = \gamma(1)$. Since S'' is compact and connected there is a nondegenerate continuum P which contains $\gamma(1) = \beta(1')$ and which is contained in $\beta(N)$. Indeed, let P be the closure of the component of $\beta(N^0)$ which contains the point $\beta(1')$. It is well known that P must meet the boundary of $\beta(N^0)$. Clearly then, $\Delta^{-1}(P)$ is a continuum which meets H' at only $L \cap H'$ and of course meets the complement of H'. Let $\Delta^{-1}(P) = Q$. Since α is an open mapping it follows from Theorem 1.5 of [3] that if K is any component of $\alpha^{-1}(Q)$ then $\alpha(K) = Q$. Letting K be such a continuum we see that K meets the complement of H and is such that $K \cap H$ is contained in some $\alpha^{-1}(n')$ where $n' \in N$. That is to say $K \cap H$ is nonvacuous and is contained in some yG for some $y \in S$, and certainly we must have $y \in H$ since $S - H$ is an ideal of S. The desired continuum may be taken as $\bar{y}K$ where $\bar{y}$ is the inverse of y in H. For if $k \in K \cap yG$ then $\bar{y}k \in G$ and if $k \notin K \cap yG$ then $k \notin H$ and $\bar{y}k \notin H$ since $S - H$ is an ideal.

COROLLARY. *If S is a compact connected semigroup with identity then the identity component of H is not a C-set.*

PROOF. One has only to note that there are arbitrarily small invariant subgroups such as G such that H/G is a Lie group.

BIBLIOGRAPHY

1. A. M. Gleason, *Spaces with a compact Lie group of transformations*, Proc. Amer. Math. Soc. vol. 1 (1950) p. 35.

2. R. J. Koch, *Note on weak cutpoints in clans*, Duke Math. J. vol. 24 (1957) p. 611.

3. G. T. Whyburn, *Interior transformations on compact sets*, Duke Math. J. vol. 3 (1957) p. 371.

OXFORD UNIVERSITY AND
THE UNIVERSITY OF GEORGIA