ON ISOMETRIC EQUIVALENCE OF CERTAIN VOLTERRA OPERATORS

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The purpose of this paper is to extend the results of §4 of the author's paper [**3**, referred to as V] to $L_p[0, 1]$ for all p such that $1<p<\infty$. In general we shall use the notation and definitions of V, except that the functions considered here are of the form

$$(1)\quad F(x, y) = (y - x)^{m-1} aG(x, y) \text{ where } \begin{cases} m \text{ is a positive integer,} \\ |a| = 1, \\ G(x, x) > 0; \end{cases}$$

otherwise, as in V, the complex valued function $G(x, y)$ is continuously differentiable. The only difference from V is the presence of the constant a which affects the proof of Theorem 2 of V. A version of that theorem in the more general case where a is an arbitrary constant of absolute value 1 will be published elsewhere [**4**]. All other theorems and proofs of V remain valid. The class D of functions with which we are principally concerned may be described as follows: the functions F are of the general form (1) where, in addition, G and m satisfy any one of the following: (A) G is analytic in a suitable region and m is an arbitrary positive integer (see Lemma 4 of V); (B) $G(x, y)=G(y-x)$ where $G(0)\neq 0$ and $G\in C^2$ in a neighborhood of $y=x$ and otherwise $G(t)\in L_1[0, 1]$ and m is an arbitrary positive integer; (C) $G\in C^2$ and $m=1$. One very important property of the operators T_F for $F\in D$ is the fact (see Theorem 3 of V) that their only reducing manifolds are the subspaces $L_p[0, c]$ of $L_p[0, 1]$ for all $c\in[0, 1]$ (see also [**2**; **5** and **6**]). This property is crucial for the establishment of unitary invariants (in the case $p=2$) of the operators T_F in §4 of V. As is usual, we define q by $1/p+1/q=1$.

Two continuous linear transformations T_1 and T_2 mapping $L_p[0, 1]$ into itself are called *isometrically equivalent* if there exists an isometry U of $L_p[0, 1]$ onto itself such that $T_1 = UT_2U^{-1}$ (regarding isometries for $p\neq 2$, see, e.g., [1, p. 178]; the considerations of the present paper are valid without this restriction). Two preliminary lemmas are needed in order to extend some results on Hilbert spaces and spectral theory to general p. We shall use the following notation: M_a is the operator "multiplication by the characteristic function $c_a(x)$ of the

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interval $[0, a]$"; similarly M_S is the operator "multiplication by the characteristic function $c_S(x)$ of the subset S of $[0, 1]$." We shall occasionally write E_a instead of M_a.

LEMMA 1. *Let T_0 be an idempotent bounded linear transformation of $L_p[0, 1]$ into itself whose range is $L_p[0, a]$ for some $a \in (0, 1]$. Then $\|T_0\| = 1$ if and only if $T_0 = M_a$.*

PROOF. Since $\|M_a\| = 1$ is obvious, we turn to the converse. Let $N_1 = L_p[0, a]$ and $N_2 = L_p[a, 1]$. Then $T_0 - M_a = T$ maps N_1 into 0 and N_2 into N_1; we have $T_0 = M_a + T$. We wish to show that $T = 0$. Suppose that $T \neq 0$; let $b = \|T\| \neq 0$. Find a positive real number e such that

$$(1 + b^q)^{1/q} - e\,\frac{b^{-1} + b^{q/p}}{(1 + b^q)^{1/p}} > 1; \tag{2}$$

now determine $f_2 \in N_2$ such that $\|Tf_2\|_p \geqq (b - e)\|f_2\|_p$ where

$$\|f_2\|_p^p = \frac{b^q}{1 + b^q}\,.$$

Let $f_1 = b^{-q} T f_2 \in N_1$ and let $f = f_1 + f_2$. Then $T_0 f = f_1 + Tf_2 = (b^{-q} + 1) Tf_2$. Since $f_i \in N_i$ $(i = 1, 2)$ we have $\|f\|_p^p = \|f_1\|_p^p + \|f_2\|_p^p$ and

$$\|f_1\|_p^p = b^{-pq}\, b^p \|f_2\|_p^p = \frac{b^{-pq} b^p b^q}{1 + b^q} = \frac{1}{1 + b^q}\,.$$

Therefore

$$\|f\|_p^p = \|f_1\|_p^p + \|f_2\|_p^p \leqq \frac{1}{1 + b^q} + \frac{b^q}{1 + b^q} = 1.$$

On the other hand,

$$\begin{aligned}
\|T_0 f\|_p &= \|f_1 + Tf_2\|_p = (b^{-q} + 1)\|Tf_2\|_p \\
&\geqq (b^{-q} + 1)(b - e)\|f_2\|_p \\
&= \frac{(b^{-q} + 1)(b - e) b^{q/p}}{(1 + b^q)^{1/p}} \\
&= \frac{(b^{-q} + 1) b^q}{(1 + b^q)^{1/p}} - e\,\frac{b^{-1} + b^{q/p}}{(1 + b^q)^{1/p}} \\
&= (1 + b^q)^{1/q} - e\,\frac{b^{-1} + b^{q/p}}{(1 + b^q)^{1/p}} \\
&> 1
\end{aligned}$$

by the choice of e, see (2). But this contradicts $\|T_0\|=1$ so that b and hence T must be zero, i.e., $T_0=M_a$, and the proof of the lemma is complete.

The following lemma shows that the projections E_a for all $a\in[0, 1]$ generate a maximal abelian algebra in the algebra of all bounded linear transformations of $L_p[0, 1]$ into itself not only for $p=2$ but indeed for all p considered in this paper.

LEMMA 2. *Let T be a bounded linear transformation mapping $L_p[0, 1]$ into itself and suppose that $TE_a=E_aT$ for all $a\in[0, 1]$. Then there exists a bounded measurable function f such that $T=M_f$ (="multiplication by f").*

PROOF. Let $e=e(x)$ be the function identically equal to 1 and let $f(x)=(Te)(x)$. We shall show that f is essentially bounded and that $T=M_f$. If g is a simple function: $g(x)=\sum_j a_jc_{S(j)}(x)$, then $Tg=\sum_j a_jTc_{S(j)}$; but $c_S(x)=(M_Se)(x)$ so that $(Tc_S)(x)=(TM_Se)(x)=(M_STe)(x)=(M_Sf)(x)=f(x)c_S(x)$ since our hypothesis implies that T commutes not only with all $E_a=M_a$ but also with all relevant M_S. Therefore $(Tg)(x)=f(x)g(x)$ for all simple g. The boundedness of T implies that $\|Tg\|_p=\|fg\|_p\leqq\|T\|_p\|g\|_p$, $\int_0^1|f(x)g(x)|^pdx\leqq\|T\|_p^p\|g\|_p^p$ for all simple g. Hence $|f|^p$ and $|f|$ are essentially bounded and $(Tg)(x)=f(x)g(x)$ for all $g\in L_p[0, 1]$.

If $s=s(t)$ is a monotone increasing function defined on $[0, 1]$ such that $s(0)=0$ and $s(1)=1$, we write $U_s=M_{(s')^{1/p}}S_s$ where we use the notation of V. If $s(t)$ is absolutely continuous with an inverse function of the same kind, then U_s as a linear transformation of $L_p[0, 1]$ into itself is an isometry onto.

THEOREM 1. *Let T_{F_1} and T_{F_2} be two continuous linear transformations of $L_p[0, 1]$ into itself whose only reducing manifolds are the subspaces $L_p[0, c]$ of $L_p[0, 1]$ for all $c\in[0, 1]$, such as, for example, the transformations T_F for $F\in D$. Then if T_{F_1} is isometrically equivalent to $T_{F_2}=UT_{F_1}U^{-1}$, there exist*: (a) *a measurable function $h(x)$ defined on $[0, 1]$ such that $|h(x)|\equiv 1$*; (b) *a strictly monotone absolutely continuous function $s(x)$ defined on $[0, 1]$ such that $s(0)=0$ and $s(1)=1$ with an inverse function of the same kind. We have $U=M_hU_s$. The functions F_1 and F_2 are then related by the equation*

$$(3)\qquad F_2(x, y)=\frac{h(x)}{h(y)}(s'(x))^{1/p}(s'(y))^{1/q}F_1(s(x), s(y)).$$

If conversely two functions F_1 and F_2 are related by (3) *where the functions $h(x)$ and $s(x)$ are defined as in* (a) *and* (b) *above, then T_{F_1} is isometrically equivalent to $T_{F_2}=UT_{F_1}U^{-1}$ where $U=M_hU_s$.*

PROOF. Suppose first that $T_{F_2}=UT_{F_1}U^{-1}$. Since both linear transformations have as their only reducing manifolds the spaces $L_p[0, c]$ for all $c\in[0, 1]$, we can conclude that $UE_tU^{-1}=F_{r(t)}$ where $F_{r(t)}$ is idempotent with range $L_p[0, r(t)]$; therefore $r(t)$ is increasing and satisfies the equations $r(0)=0$ and $r(1)=1$. Since $\|E_t\|=1$ and U is an isometry, we have $\|F_{r(t)}\|=1$ for all positive $r(t)$. Lemma 1 now implies that $F_{r(t)}=E_{r(t)}=UE_tU^{-1}$. To show that r is absolutely continuous and strictly increasing, we consider for $g\in L_q[0, 1]$ the expression $(E_{r(t)}f, g)=(UE_tU^{-1}f, g)=(E_tf_1, g_1)$ where $f_1=U^{-1}f$ and $g_1=U^*g$; the linear transformation U^* is the adjoint of U (acting in $L_q[0, 1]$). If $f=g=1$ then $r(t)=\int_0^t f_1(x)g_1(x)dx$. This shows that $r(t)$ is absolutely continuous. If $f_1=g_1=1$ then $t=\int_0^{(r)t}f(s)g(x)dx$. This shows that $r(t)$ is strictly increasing; the inverse function $s(t)$ of $r(t)$ has the same properties. It is easy to verify that $U_r^{-1}E_tU_r=E_{r(t)}$; this equation together with the equation $UE_tU^{-1}=E_{r(t)}$ implies that U_rU commutes with all E_t. Lemma 2 then implies that $U_rU=M_k$ where $k=k(x)$ is a bounded measurable function. Since U_rU is an isometry, the function $k(x)$ must satisfy $|k(x)| \equiv 1$; we have $U=U_r^{-1}M_k$. The functions $s(x)$ and $h(x)=k(s(x))$ are the functions whose existence was asserted by the theorem; a simple calculation shows that $U=U_r^{-1}M_k=M_hU_s$ as promised. It is now an easy matter to verify (3); the computation needed for this purpose is similar to that needed to establish the converse of the theorem.

We state next the analog of Theorem 5 of V; the formulas and proof are changed due to the presence of the constant a in our present context, and to the arbitrariness of p.

THEOREM 2. *Let $F(x, y)=(y-x)^{m-1}\,a\,G(x, y)$ be of form* (1) *where $G\in C^1$ in a neighborhood of $y=x$ and let T_F, considered as a linear transformation of $L_p[0, 1]$ into itself, have as its only reducing manifolds the spaces $L_p[0, c]$ for all $c\in[0, 1]$, as is the case if $F\in D$. Then T_F is isometrically equivalent to a unique $T_{F_1}=UT_FU^{-1}$ where F_1 and G_1 satisfy the following*:

$$
\begin{aligned}
&F_1(x, y) = (y-x)^{m-1}aG_1(x, y),\\
(4)\qquad &G_1(x, x) = c = \left(\int_0^1 (G(u, u))^{1/m}du\right)^m > 0,\\
&\mathrm{Im}\,(G_{1x}(x, x)) = \mathrm{Im}\,(G_{1y}(x, x)) = 0.
\end{aligned}
$$

This is achieved by setting $U=M_hU_r^{-1}$ where

$$r(t) = (1/c)^{1/m}\int_0^t (G(u, u))^{1/m}du.$$

The function $h(x)$ is determined by defining

$$F_0(x, y) = (y - x)^{m-1} a G_0(x, y)$$

by $T_{F_0} = U_r^{-1} T_F U_r$ where $G_0 = H_0 + iK_0$ for real H_0 and K_0 and setting $h(x) = \exp((-i/c)\int_0^x K_{0x}(u, u)du)$.

Proof. If r and h are defined as described and if we set $T_{F_1} = (M_h U_r^{-1}) T_F (M_h U_r^{-1})^{-1}$ then F_1 does indeed satisfy (4). Thus every T_F is isometric with T_{F_1}, where F_1 has form (4). To show uniqueness, suppose that $F_i = (y-x)^{m_i-1} a_i G_i$ $(i=1, 2)$ are both of form (4) and that T_{F_1} is isometrically equivalent to $T_{F_2} = U T_{F_1} U^{-1}$. Then (3) implies that F_1 and F_2 are related by the following equation:

$$(y - x)^{m_2-1} a_2 G_2(x, y)$$
$$= \frac{h(x)}{h(y)} (s'(x))^{1/p} (s'(y))^{1/q} \left(\frac{s(y) - s(x)}{y - x}\right)^{m_1-1} (y - x)^{m_1-1} a_1 G_1(s(x), s(y)),$$

where the functions h and s are as in (a) and (b) of Theorem 2. On letting $y-x$ approach zero we see that $m_1 = m_2 = m$ and that

$$a_2 G_2(x, x) = (s'(x))^m a_1 G_1(s(x), s(x)) \tag{5}$$

since $1/p + 1/q + m - 1 = m$. Equation (5) implies that $a_1 = a_2 = a$ since $G_i(x, x) = c_i > 0$; we next observe that (5) also implies that $s(x) \equiv x$ and that $c_1 = c_2 = c$. Equation (3) now reduces to $G_2(x, y) = (h(x)/h(y)) G_1(x, y)$ or

$$h(y)(H_2(x, y) + iK_2(x, y)) = h(x)(H_1(x, y) + iK_1(x, y)), \tag{6}$$

if we write $G_j = H_j + iK_j$ for real H_j and K_j $(j=1, 2)$. Our hypotheses imply that $H_j(x, x) = c$ and that $K_j(x, x) = K_{jx}(x, x) = K_{jy}(x, x) = 0$ $(j=1, 2)$ and also that $h(x) = \exp(ik(x))$ for real $k(x)$ is differentiable. Differentiation of (6) and setting $x = y$ yields $h(x) H_{2x}(x, x) = ch'(x) + h(x) H_{1x}(x, x)$ so that $h'/h = ik' = 1/c(H_{2x}(x, x) - H_{1x}(x, x))$. But the last expression is real so that $k' = 0$, and h is constant. We finally arrive at $G_1 = G_2$: *If two functions F_1 and F_2 satisfy* (4) *and if the corresponding operators T_{F_1} and T_{F_2} are isometrically equivalent and have the spaces $L_p[0, c]$ for all $c \in [0, 1]$ as their only reducing manifolds—for example, if the functions $F_j \in D$—then $F_1 = F_2$.*

Observe that if our functions F belong to D, then the similarity invariants of T_F, viz., m, a, and c, enter directly into the formulation of the isometry invariants (see V and [4] for similarity invariants). The "canonical functions" F_1 as given by (4) are the same for all p; however a given T_F will have as its "canonical form" T_{F_1} a transformation which in general *does* depend on p. If, for example, $F(x, y)$

$=1+2x+i(x-y)$, then $m=1$, $a=1$, $c=2$. To describe its "canonical form" F_1 satisfying (4), it is convenient to introduce the function $K(x,\ y)=((8x+1)/(8y+1))^{1/2}$. A simple calculation shows that $F_1(x,\ y)=2\exp(-i\log K)(K^{1/q}+i(K^{1/q}-K^{-1/p}))$.

References

1. S. Banach, *Théorie des opérations linéaires*, Warszawa, 1932.
2. M. S. Brodskiĭ and M. S. Livšic, *Spektral'nyi analiz nesamosopryažennyh operatorov i promežutočnye sistemy*, Uspehi Mat. Nauk vol. 13 (1958) pp. 3–85.
3. G. K. Kalisch, *On similarity, reducing manifolds, and unitary equivalence of certain Volterra operators*, Ann. of Math. vol. 66 (1957) pp. 481–494.
4. ———, *On similarity invariants of certain operators in L_p*, Pacific J. Math., to appear.
5. L. A. Sahnovič, *O privedenii vol'terrovskih operatorov k prosteĭšemu vidu i obratnyh sadačah*, Izv. Akad. Nauk SSSR. Ser. Mat. vol. 21 (1957) pp. 235–262.
6. ———, *Spektral'nyi analiz operatorov vida $Kf=\int_0^x f(t)k(x-t)dt$*, Izv. Akad. Nauk SSSR. Ser. Mat. vol. 22 (1958) pp. 299–308.

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THE NONEXISTENCE OF PROJECTIONS FROM L^1 TO H^1

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Consider the Banach space $L^1(0,\ 2\pi)$ and the subspace H^1, of all functions all of whose negative Fourier coefficients vanish. It has been conjectured that H^1 has no complement in L^1, i.e., that L^1 is not the direct sum of H^1 and some other Banach space. In this note we give a proof of this conjecture.

The conjecture is of course equivalent to the following statement on projection operators.

Theorem. *There exists no bounded linear operator $P: L^1 \to H^1$ for which $Pf=f$ for all $f \in H^1$.*

Proof. Suppose such a P existed. Let $l_n(f)$ denote the nth Fourier coefficient of $P(f)$; then l_n is a bounded linear functional on L^1 and as a result we have

$$l_n(f(\theta)) = \int_0^{2\pi} f(\theta)\phi_n(\theta)d\theta,$$

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