# ON ISOMETRIC EQUIVALENCE OF CERTAIN VOLTERRA OPERATORS

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The purpose of this paper is to extend the results of §4 of the author's paper [3, referred to as V] to $L_p[0, 1]$ for all $p$ such that $1<p<\infty$. In general we shall use the notation and definitions of V, except that the functions considered here are of the form

$$(1)\quad F(x, y) = (y - x)^{m-1}aG(x, y) \text{ where } \begin{cases} m \text{ is a positive integer,} \\ |a| = 1, \\ G(x, x) > 0; \end{cases}$$

otherwise, as in V, the complex valued function $G(x, y)$ is continuously differentiable. The only difference from V is the presence of the constant $a$ which affects the proof of Theorem 2 of V. A version of that theorem in the more general case where $a$ is an arbitrary constant of absolute value 1 will be published elsewhere [4]. All other theorems and proofs of V remain valid. The class $D$ of functions with which we are principally concerned may be described as follows: the functions $F$ are of the general form (1) where, in addition, $G$ and $m$ satisfy any one of the following: (A) $G$ is analytic in a suitable region and $m$ is an arbitrary positive integer (see Lemma 4 of V); (B) $G(x, y)=G(y-x)$ where $G(0)\neq 0$ and $G\in C^2$ in a neighborhood of $y=x$ and otherwise $G(t)\in L_1[0, 1]$ and $m$ is an arbitrary positive integer; (C) $G\in C^2$ and $m=1$. One very important property of the operators $T_F$ for $F\in D$ is the fact (see Theorem 3 of V) that their only reducing manifolds are the subspaces $L_p[0, c]$ of $L_p[0, 1]$ for all $c\in[0, 1]$ (see also [2; 5 and 6]). This property is crucial for the establishment of unitary invariants (in the case $p=2$) of the operators $T_F$ in §4 of V. As is usual, we define $q$ by $1/p+1/q=1$.

Two continuous linear transformations $T_1$ and $T_2$ mapping $L_p[0, 1]$ into itself are called *isometrically equivalent* if there exists an isometry $U$ of $L_p[0, 1]$ onto itself such that $T_1=UT_2U^{-1}$ (regarding isometries for $p\neq 2$, see, e.g., [1, p. 178]; the considerations of the present paper are valid without this restriction). Two preliminary lemmas are needed in order to extend some results on Hilbert spaces and spectral theory to general $p$. We shall use the following notation: $M_a$ is the operator "multiplication by the characteristic function $c_a(x)$ of the

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interval $[0, a]$"; similarly $M_S$ is the operator "multiplication by the characteristic function $c_S(x)$ of the subset $S$ of $[0, 1]$." We shall occasionally write $E_a$ instead of $M_a$.

LEMMA 1. *Let $T_0$ be an idempotent bounded linear transformation of $L_p[0, 1]$ into itself whose range is $L_p[0, a]$ for some $a \in (0, 1]$. Then $\|T_0\| = 1$ if and only if $T_0 = M_a$.*

PROOF. Since $\|M_a\| = 1$ is obvious, we turn to the converse. Let $N_1 = L_p[0, a]$ and $N_2 = L_p[a, 1]$. Then $T_0 - M_a = T$ maps $N_1$ into 0 and $N_2$ into $N_1$; we have $T_0 = M_a + T$. We wish to show that $T = 0$. Suppose that $T \neq 0$; let $b = \|T\| \neq 0$. Find a positive real number $e$ such that

$$(1 + b^q)^{1/q} - e\frac{b^{-1} + b^{q/p}}{(1 + b^q)^{1/p}} > 1; \tag{2}$$

now determine $f_2 \in N_2$ such that $\|Tf_2\|_p \geqq (b - e)\|f_2\|_p$ where

$$\|f_2\|_p^p = \frac{b^q}{1 + b^q}.$$

Let $f_1 = b^{-q}Tf_2 \in N_1$ and let $f = f_1 + f_2$. Then $T_0 f = f_1 + Tf_2 = (b^{-q} + 1)Tf_2$. Since $f_i \in N_i$ $(i = 1, 2)$ we have $\|f\|_p^p = \|f_1\|_p^p + \|f_2\|_p^p$ and

$$\|f_1\|_p^p = b^{-pq}\, b^p \|f_2\|_p^p = \frac{b^{-pq}b^p b^q}{1 + b^q} = \frac{1}{1 + b^q}.$$

Therefore

$$\|f\|_p^p = \|f_1\|_p^p + \|f_2\|_p^p \leqq \frac{1}{1 + b^q} + \frac{b^q}{1 + b^q} = 1.$$

On the other hand,

$$\begin{aligned}
\|T_0 f\|_p &= \|f_1 + Tf_2\|_p = (b^{-q} + 1)\|Tf_2\|_p \\
&\geqq (b^{-q} + 1)(b - e)\|f_2\|_p \\
&= \frac{(b^{-q} + 1)(b - e)b^{q/p}}{(1 + b^q)^{1/p}} \\
&= \frac{(b^{-q} + 1)b^q}{(1 + b^q)^{1/p}} - e\frac{b^{-1} + b^{q/p}}{(1 + b^q)^{1/p}} \\
&= (1 + b^q)^{1/q} - e\frac{b^{-1} + b^{q/p}}{(1 + b^q)^{1/p}} \\
&> 1
\end{aligned}$$

by the choice of $e$, see (2). But this contradicts $\|T_0\|=1$ so that $b$ and hence $T$ must be zero, i.e., $T_0=M_a$, and the proof of the lemma is complete.

The following lemma shows that the projections $E_a$ for all $a\in[0, 1]$ generate a maximal abelian algebra in the algebra of all bounded linear transformations of $L_p[0, 1]$ into itself not only for $p=2$ but indeed for all $p$ considered in this paper.

LEMMA 2. *Let $T$ be a bounded linear transformation mapping $L_p[0, 1]$ into itself and suppose that $TE_a=E_aT$ for all $a\in[0, 1]$. Then there exists a bounded measurable function $f$ such that $T=M_f$ (= "multiplication by $f$").*

PROOF. Let $e=e(x)$ be the function identically equal to 1 and let $f(x)=(Te)(x)$. We shall show that $f$ is essentially bounded and that $T=M_f$. If $g$ is a simple function: $g(x)=\sum_j a_j c_{S(j)}(x)$, then $Tg=\sum_j a_j Tc_{S(j)}$; but $c_S(x)=(M_Se)(x)$ so that $(Tc_S)(x)=(TM_Se)(x)=(M_STe)(x)=(M_Sf)(x)=f(x)c_S(x)$ since our hypothesis implies that $T$ commutes not only with all $E_a=M_a$ but also with all relevant $M_S$. Therefore $(Tg)(x)=f(x)g(x)$ for all simple $g$. The boundedness of $T$ implies that $\|Tg\|_p=\|fg\|_p\leqq\|T\|_p\|g\|_p$, $\int_0^1|f(x)g(x)|^p dx\leqq\|T\|_p^p\|g\|_p^p$ for all simple $g$. Hence $|f|^p$ and $|f|$ are essentially bounded and $(Tg)(x)=f(x)g(x)$ for all $g\in L_p[0, 1]$.

If $s=s(t)$ is a monotone increasing function defined on $[0, 1]$ such that $s(0)=0$ and $s(1)=1$, we write $U_s=M_{(s')^{1/p}}S_s$ where we use the notation of V. If $s(t)$ is absolutely continuous with an inverse function of the same kind, then $U_s$ as a linear transformation of $L_p[0, 1]$ into itself is an isometry onto.

THEOREM 1. *Let $T_{F_1}$ and $T_{F_2}$ be two continuous linear transformations of $L_p[0, 1]$ into itself whose only reducing manifolds are the subspaces $L_p[0, c]$ of $L_p[0, 1]$ for all $c\in[0, 1]$, such as, for example, the transformations $T_F$ for $F\in D$. Then if $T_{F_1}$ is isometrically equivalent to $T_{F_2}=UT_{F_1}U^{-1}$, there exist:* (a) *a measurable function $h(x)$ defined on $[0, 1]$ such that $|h(x)|\equiv 1$;* (b) *a strictly monotone absolutely continuous function $s(x)$ defined on $[0, 1]$ such that $s(0)=0$ and $s(1)=1$ with an inverse function of the same kind. We have $U=M_hU_s$. The functions $F_1$ and $F_2$ are then related by the equation*

$$F_2(x, y) = \frac{h(x)}{h(y)}(s'(x))^{1/p}(s'(y))^{1/q}F_1(s(x), s(y)). \tag{3}$$

*If conversely two functions $F_1$ and $F_2$ are related by* (3) *where the functions $h(x)$ and $s(x)$ are defined as in* (a) *and* (b) *above, then $T_{F_1}$ is isometrically equivalent to $T_{F_2}=UT_{F_1}U^{-1}$ where $U=M_hU_s$.*

PROOF. Suppose first that $T_{F_2}=UT_{F_1}U^{-1}$. Since both linear transformations have as their only reducing manifolds the spaces $L_p[0, c]$ for all $c\in[0, 1]$, we can conclude that $UE_tU^{-1}=F_{r(t)}$ where $F_{r(t)}$ is idempotent with range $L_p[0, r(t)]$; therefore $r(t)$ is increasing and satisfies the equations $r(0)=0$ and $r(1)=1$. Since $\|E_t\|=1$ and $U$ is an isometry, we have $\|F_{r(t)}\|=1$ for all positive $r(t)$. Lemma 1 now implies that $F_{r(t)}=E_{r(t)}=UE_tU^{-1}$. To show that $r$ is absolutely continuous and strictly increasing, we consider for $g\in L_q[0, 1]$ the expression $(E_{r(t)}f, g)=(UE_tU^{-1}f, g)=(E_tf_1, g_1)$ where $f_1=U^{-1}f$ and $g_1=U^*g$; the linear transformation $U^*$ is the adjoint of $U$ (acting in $L_q[0, 1]$). If $f=g=1$ then $r(t)=\int_0^t f_1(x)g_1(x)dx$. This shows that $r(t)$ is absolutely continuous. If $f_1=g_1=1$ then $t=\int_0^{(r)t}f(s)g(x)dx$. This shows that $r(t)$ is strictly increasing; the inverse function $s(t)$ of $r(t)$ has the same properties. It is easy to verify that $U_r^{-1}E_tU_r=E_{r(t)}$; this equation together with the equation $UE_tU^{-1}=E_{r(t)}$ implies that $U_rU$ commutes with all $E_t$. Lemma 2 then implies that $U_rU=M_k$ where $k=k(x)$ is a bounded measurable function. Since $U_rU$ is an isometry, the function $k(x)$ must satisfy $|k(x)| \equiv 1$; we have $U=U_r^{-1}M_k$. The functions $s(x)$ and $h(x)=k(s(x))$ are the functions whose existence was asserted by the theorem; a simple calculation shows that $U=U_r^{-1}M_k=M_hU_s$ as promised. It is now an easy matter to verify (3); the computation needed for this purpose is similar to that needed to establish the converse of the theorem.

We state next the analog of Theorem 5 of V; the formulas and proof are changed due to the presence of the constant $a$ in our present context, and to the arbitrariness of $p$.

THEOREM 2. *Let $F(x, y)=(y-x)^{m-1}\, a\, G(x, y)$ be of form* (1) *where $G\in C^1$ in a neighborhood of $y=x$ and let $T_F$, considered as a linear transformation of $L_p[0, 1]$ into itself, have as its only reducing manifolds the spaces $L_p[0, c]$ for all $c\in[0, 1]$, as is the case if $F\in D$. Then $T_F$ is isometrically equivalent to a unique $T_{F_1}=UT_FU^{-1}$ where $F_1$ and $G_1$ satisfy the following*:

$$F_1(x, y) = (y - x)^{m-1}aG_1(x, y),$$

$$G_1(x, x) = c = \left(\int_0^1 (G(u, u))^{1/m}du\right)^m > 0, \tag{4}$$

$$\operatorname{Im}(G_{1x}(x, x)) = \operatorname{Im}(G_{1y}(x, x)) = 0.$$

This is achieved by setting $U=M_hU_r^{-1}$ where

$$r(t) = (1/c)^{1/m}\int_0^t (G(u, u))^{1/m}du.$$

The function $h(x)$ is determined by defining

$$F_0(x, y) = (y - x)^{m-1}aG_0(x, y)$$

by $T_{F_0} = U_r^{-1}T_F U_r$ where $G_0 = H_0 + iK_0$ for real $H_0$ and $K_0$ and setting $h(x) = \exp((-i/c)\int_0^x K_{0x}(u, u)du)$.

PROOF. If $r$ and $h$ are defined as described and if we set $T_{F_1} = (M_h U_r^{-1})T_F(M_h U_r^{-1})^{-1}$ then $F_1$ does indeed satisfy (4). Thus every $T_F$ is isometric with $T_{F_1}$, where $F_1$ has form (4). To show uniqueness, suppose that $F_i = (y-x)^{m_i-1}a_iG_i$ $(i=1, 2)$ are both of form (4) and that $T_{F_1}$ is isometrically equivalent to $T_{F_2} = UT_{F_1}U^{-1}$. Then (3) implies that $F_1$ and $F_2$ are related by the following equation:

$$(y - x)^{m_2-1}a_2G_2(x, y)$$
$$= \frac{h(x)}{h(y)}(s'(x))^{1/p}(s'(y))^{1/q}\left(\frac{s(y) - s(x)}{y - x}\right)^{m_1-1}(y - x)^{m_1-1}a_1G_1(s(x), s(y)),$$

where the functions $h$ and $s$ are as in (a) and (b) of Theorem 2. On letting $y-x$ approach zero we see that $m_1 = m_2 = m$ and that

$$a_2G_2(x, x) = (s'(x))^m a_1G_1(s(x), s(x)) \tag{5}$$

since $1/p + 1/q + m - 1 = m$. Equation (5) implies that $a_1 = a_2 = a$ since $G_i(x, x) = c_i > 0$; we next observe that (5) also implies that $s(x) \equiv x$ and that $c_1 = c_2 = c$. Equation (3) now reduces to $G_2(x, y) = (h(x)/h(y))G_1(x, y)$ or

$$h(y)(H_2(x, y) + iK_2(x, y)) = h(x)(H_1(x, y) + iK_1(x, y)), \tag{6}$$

if we write $G_j = H_j + iK_j$ for real $H_j$ and $K_j$ $(j=1, 2)$. Our hypotheses imply that $H_j(x, x) = c$ and that $K_j(x, x) = K_{jx}(x, x) = K_{jy}(x, x) = 0$ $(j=1, 2)$ and also that $h(x) = \exp(ik(x))$ for real $k(x)$ is differentiable. Differentiation of (6) and setting $x=y$ yields $h(x)H_{2x}(x, x) = ch'(x) + h(x)H_{1x}(x, x)$ so that $h'/h = ik' = 1/c(H_{2x}(x, x) - H_{1x}(x, x))$. But the last expression is real so that $k' = 0$, and $h$ is constant. We finally arrive at $G_1 = G_2$: *If two functions $F_1$ and $F_2$ satisfy* (4) *and if the corresponding operators $T_{F_1}$ and $T_{F_2}$ are isometrically equivalent and have the spaces $L_p[0, c]$ for all $c \in [0, 1]$ as their only reducing manifolds—for example, if the functions $F_j \in D$—then $F_1 = F_2$.*

Observe that if our functions $F$ belong to $D$, then the similarity invariants of $T_F$, viz., $m$, $a$, and $c$, enter directly into the formulation of the isometry invariants (see V and [4] for similarity invariants). The "canonical functions" $F_1$ as given by (4) are the same for all $p$; however a given $T_F$ will have as its "canonical form" $T_{F_1}$ a transformation which in general *does* depend on $p$. If, for example, $F(x, y)$

$=1+2x+i(x-y)$, then $m=1$, $a=1$, $c=2$. To describe its "canonical form" $F_1$ satisfying (4), it is convenient to introduce the function $K(x,\ y)=((8x+1)/(8y+1))^{1/2}$. A simple calculation shows that $F_1(x,\ y)=2\ \exp(-i\log K)(K^{1/q}+i(K^{1/q}-K^{-1/p}))$.

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# THE NONEXISTENCE OF PROJECTIONS FROM $L^1$ TO $H^1$

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Consider the Banach space $L^1(0,\ 2\pi)$ and the subspace $H^1$, of all functions all of whose negative Fourier coefficients vanish. It has been conjectured that $H^1$ has no complement in $L^1$, i.e., that $L^1$ is not the direct sum of $H^1$ and some other Banach space. In this note we give a proof of this conjecture.

The conjecture is of course equivalent to the following statement on projection operators.

Theorem. *There exists no bounded linear operator $P: L^1\to H^1$ for which $Pf=f$ for all $f\in H^1$.*

Proof. Suppose such a $P$ existed. Let $l_n(f)$ denote the $n$th Fourier coefficient of $P(f)$; then $l_n$ is a bounded linear functional on $L^1$ and as a result we have

$$l_n(f(\theta)) = \int_0^{2\pi} f(\theta)\phi_n(\theta)d\theta,$$

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