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\[ = 1 + 2x + i(x - y), \text{ then } m = 1, a = 1, c = 2. \text{ To describe its “canonical form” } F_1 \text{ satisfying (4), it is convenient to introduce the function } K(x, y) = \left( \frac{(8x+1)/(8y+1)}{2} \right)^{1/2}. \text{ A simple calculation shows that } F_1(x, y) = 2 \exp(-i \log K)(K^{1/2} + i(K^{1/2} - K^{-1/2})). \]

REFERENCES


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THE NONEXISTENCE OF PROJECTIONS FROM \( L^1 \) TO \( H^1 \)

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Consider the Banach space \( L^1(0, 2\pi) \) and the subspace \( H^1 \), of all functions all of whose negative Fourier coefficients vanish. It has been conjectured that \( H^1 \) has no complement in \( L^1 \), i.e., that \( L^1 \) is not the direct sum of \( H^1 \) and some other Banach space. In this note we give a proof of this conjecture.

The conjecture is of course equivalent to the following statement on projection operators.

**Theorem.** There exists no bounded linear operator \( P: L^1 \to H^1 \) for which \( Pf = f \) for all \( f \in H^1 \).

**Proof.** Suppose such a \( P \) existed. Let \( l_n(f) \) denote the \( n \)th Fourier coefficient of \( P(f) \); then \( l_n \) is a bounded linear functional on \( L^1 \) and as a result we have

\[
l_n(f(\theta)) = \int_0^{2\pi} f(\theta)\phi_n(\theta)\,d\theta,
\]

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where $\phi_n(\theta)$ is a bounded function. Also we know that

$$l_n(e^{nt}) = 1 \quad \text{(assuming } n \geq 0)$$

and so

$$\int_0^{2\pi} |\phi_n(\theta)| \, d\theta \geq 1.$$ 

It follows that

$$\int_0^{2\pi} \sum_{n=1}^{N} \frac{|\phi_n(\theta)|}{n} \, d\theta \geq \log N, \quad N > 1, \text{ } N \text{ fixed.}$$

So that, for some $\theta_0$, $\sum_{n=1}^{N} |\phi_n(\theta_0)|/n \geq (1/2\pi) \log N$. Thus choosing $\varepsilon_n = \text{sg } \phi_n(\theta_0)$ we are assured that $\max_\theta \left| \sum_{n=1}^{N} \varepsilon_n \phi_n(\theta)/n \right| \geq (1/2\pi) \log N$ (the inequality holding in fact for $\theta_0$).

We can therefore determine a function $f(\theta)$ with $\int_0^{2\pi} |f| \leq 1$ such that

$$\int_0^{2\pi} f(\theta) \sum_{n=1}^{N} \frac{\varepsilon_n \phi_n(\theta)}{n} \, d\theta \geq \frac{1}{10} \log N$$

but

$$\left| \int_0^{2\pi} f(\theta) \sum_{n=1}^{N} \frac{\varepsilon_n \phi_n(\theta)}{n} \, d\theta \right| \leq \sum_{n=1}^{N} \frac{|l_n(f)|}{n} = \sum_{n=1}^{N} \frac{|a_n|}{n},$$

where $\sum a_n e^{i n \theta}$ is the Fourier series for $Pf$. Since $Pf \in H^1$, however, by Hardy's theorem [1],

$$\sum_{n=1}^{N} \frac{|a_n|}{n} \leq 2\pi \int |Pf| \leq 2\pi \|P\|.$$ 

Finally combining (1) and (2) gives the contradiction

$$\frac{1}{10} \log N \leq 2\pi \|P\|,$$

for all $N > 1$,

and this completes the proof.

Reference


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