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AN EXTENSION OF BANACH'S CONTRACTION PRINCIPLE

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1. Let X be a metric space and f a mapping of X into itself; f will be said to be a globally contractive mapping if the condition

$$d(f(p), f(q)) < \lambda d(p, q) \tag{1}$$

with constant λ, $0 \leqq \lambda < 1$, holds for every p, $q \in X$. A well known theorem of Banach states:

If X is a complete metric space and f is a globally contractive mapping of X into itself then there exists a unique point ξ such that $f(\xi) = \xi$.

2. It is natural to ask whether the theorem (referred to as Banach's contraction principle) could be modified so as to be valid when condition (1) is assumed to hold for sufficiently close points only. To be more specific we introduce the following notions:

2.1. A mapping f of X into itself is said to be locally contractive if for every $x \in X$ there exist ϵ and λ ($\epsilon > 0$, $0 \leqq \lambda < 1$), which may depend on x, such that:

$$p, q \in S(x, \epsilon) = \{y \mid d(x, y) < \epsilon\} \quad \text{implies (1).} \tag{2}$$

2.2. A mapping f of X into itself is said to be (ϵ, λ)-uniformly

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locally contractive if it is locally contractive and both ϵ and λ do not depend on x.

We remark that a globally contractive mapping can be regarded as a (∞, λ) uniformly locally contractive mapping.

3. For some special spaces every locally contractive mapping is globally contractive. We prove namely the following

PROPOSITION. *If X is a convex, complete, metric space then every mapping f of X into itself which is (ϵ, λ)-uniformly contractive is also globally contractive with the same λ.*

The convexity in the above assertion is to be understood in the sense of Menger. A subset $M\subset X$ is, accordingly, convex if for every pair a, $b\in M$ there exists a point $c\in M$ such that $d(a, b)=d(a, c)+d(c, b)$.

A theorem by Menger [1, p. 41] states that a convex and complete metric space contains together with a, b also a metric segment whose extremities are a and b; that is a subset isometric to an interval of length $d(a, b)$.

Using this fact we see that if p, $q\in M$ then there are points $p=x_0, x_1, \cdots, x_n=q$ such that $d(p, q)=\sum_1^n d(x_{i-1}, x_i)$ and $d(x_{i-1}, x_i)<\epsilon$.

Hence: $d(f(p),f(q))\leqq \sum_1^n d(f(x_{i-1}),f(x_i))<\lambda \sum_1^n d(x_{i-1},x_i)=\lambda d(p,q)$ as asserted.

4. On the other hand, it is quite easy to exhibit spaces which admit locally contractive or even uniformly locally contractive mappings which are not globally contractive. The following is a simple example:

$$X = \left\{(x, y) \mid x = \cos t, y = \sin t, 0 \leqq t \leqq \frac{3}{2}\pi\right\}; \qquad f(t) = \frac{t}{2} \cdot$$

X is taken in the metric of the euclidean plane.

5. **Extended contraction principle.** 5.1. A metric space X will be said to be η-chainable if for every a, $b\in X$ there exists an η-chain, that is a finite set of points $a=x_0, x_1, \cdots, x_n=b$ (n may depend on both a and b) such that $d(x_{i-1}, x_i)<\eta$ $(i=1, 2, \cdots, n)$.

5.2. THEOREM. *Let X be a complete metric ϵ-chainable space, f a mapping of X into itself which is (ϵ, λ)-uniformly locally contractive then there exists a unique point $\xi\in X$ such that $f(\xi)=\xi$.*

PROOF. Let x be an arbitrary point of X. Consider the ϵ-chain:

$$x = x_0, x_1, \cdots, x_n = f(x);$$

by the triangle inequality: $d(x, f(x)) \leqq \sum_1^n d(x_{i-1}, x_i) < n\epsilon$.

For pairs of consecutive points of the ϵ-chain, condition (2) is satisfied.

Hence, denoting $f(f^m(x)) = f^{m+1}(x)$ $(m=1, 2, \cdots)$ we have:

$$d(f(x_{i-1}), f(x_i)) < \lambda d(x_{i-1}, x_i) < \lambda\epsilon;$$

and, by induction:

$$d(f^m(x_{i-1}), f^m(x_i)) < \lambda d(f^{m-1}(x_{i-1}), f^{m-1}(x_i)) < \lambda^m\epsilon \cdots. \tag{3}$$

From the last inequality we obtain:

$$d(f^m(x), f^{m+1}(x)) \leqq \sum_{i=1}^{n} d(f^m(x_{i-1}), f^m(x_i)) < \lambda^m n\epsilon.$$

It follows that the sequence of iterates $\{f^i(x)\}$ is a Cauchy sequence.

Indeed if j and k $(j<k)$ are positive integers then:

$$d(f^j(x), f^k(x)) \leqq \sum_{i=j}^{k-1} d(f^i(x), f^{i+1}(x)) < n\epsilon\,(\lambda^j + \cdots + \lambda^{k-1})$$

$$< n\epsilon\frac{\lambda^j}{1-\lambda} \to 0, \qquad j \uparrow \infty.$$

The completeness of X guarantees the existence of $\lim_{i\to\infty} f^i(x)$.

From the continuity of f (clearly implied by (2)) it then follows that:

$$f\left(\lim_{i\to\infty} f^i(x)\right) = \lim_{i\to\infty} f(f^i(x)) = \lim_{i\to\infty} f^{i+1}(x) = \lim_{i\to\infty} f^i(x).$$

The proof will now be completed if it is shown that $\xi = \lim_{i\to\infty} f^i(x)$ is the only point satisfying $f(\xi)=\xi$. Suppose, then, that there exists $\xi' \neq \xi$ (hence $d(\xi, \xi') > 0$) with the property $\xi' = f(\xi')$ and let $\xi = x_0, x_1, \cdots, x_k = \xi'$ be an ϵ-chain. Using (3) again we obtain:

$$\begin{aligned} d(f(\xi), f(\xi')) &= d(f^l(\xi), f^l(\xi')) \\ &\leqq \sum_{i=1}^{k} d(f^l(x_{i-1}), f^l(x_i)) < \lambda^l k\epsilon \to 0, \qquad l \uparrow \infty \end{aligned}$$

which is impossible. Hence $\xi = \xi'$ and our proof is completed.

6. From the above theorem there follows a corollary regarding expansive mappings. These mappings can be defined in a natural way by replacing $\lambda < 1$ with $\lambda > 1$ in 2.1 and 2.2. Hence we obtain:

6.1. COROLLARY. *If f is a one-to-one (ϵ, λ)-uniformly locally expan-*

sive mapping of a metric space Y onto an ϵ-chainable complete metric space $X\supset Y$ then there exists a unique ξ such that $f(\xi)=\xi$.

This assertion is an immediate consequence of the fact that for the inverse mapping $f^{-1}(x)$ all assumptions of the theorem are satisfied.

7. An application to analytic functions of a complex variable.

7.1. PROPOSITION. *Let $f(z)$ be an analytic function in a domain D of the complex z-plane; let $f(z)$ map a compact and connected subset C of D into itself. If, in addition, $|f'(z)|<1$ for every $z\in C$ then the equation $f(z)=z$ has one and only one solution in C.*

PROOF. As $|f'(z)|$ is continuous on C it follows from the compactness of C that $|f'(z)|<\lambda<1$ on C. To prove our proposition it suffices to show that there exists an $\epsilon>0$, such that f is (ϵ, λ)-uniformly locally contractive on C.

To this end consider a cover of C by a family of open discs $\{S(z, \rho)\}$, centered at points $z\in C$ and of radius ρ, such that $f(z)$ is analytic and $|f'(z)|<\lambda$ in $S(z, 2\rho)$. This cover contains, again by the compactness of C, a finite subcover $\{S(z_i, \rho_i)\}$ $(i=1, 2, \cdots, n)$.

Put $\epsilon=\min_i \rho_i$. Any two points of C distant less then ϵ will, evidently, fall into some $S(z_j, 2\rho_j)$. Hence

$$|f(z)-f(z')| = \left|\int_{z'}^{z} f'(z)dz\right| < \lambda|z-z'| \quad (z, z'\in C, \ |z-z'| < \epsilon),$$

and Theorem 5.2 applies.

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