SOLUTIONS OF LINEAR DIFFERENTIAL SYSTEMS SATISFYING BOUNDARY CONDITIONS IN THE LARGE

J. B. GARNER AND L. P. BURTON

1. Introduction. In recent years existence and uniqueness theorems have been given for differential systems where multiple-point boundary conditions are imposed. For those theorems which apply to the linear system

$$(1) \qquad y_i' = \sum_{j=1}^{n} a_{ij}(x)y_j + b_i(x), \qquad i = 1, \cdots, n,$$

the interval over which the boundary points are distributed is restricted in length. In the present paper conditions on the $a_{ij}(x)$ are given which assure a unique solution satisfying two-point and three-point boundary conditions where these points are required only to belong to the interval, say $[a, b]$, over which the $a_{ij}(x)$, $b_i(x)$ are continuous.

2. Two-point boundary conditions. For points α_1, α_2, $\alpha_1<\alpha_2$, of $[a, b]$ we define the following conditions over $[\alpha_1, \alpha_2]$:

A. $a_{ij}(x)$, $i\neq j$, is nonzero.

B. If $a_{mn}(x)>0$, $a_{mk}(x)$ has the same sign as $a_{kn}(x)$; if $a_{mn}(x)<0$, $a_{mk}(x)$ has the opposite sign to $a_{kn}(x)$; $a_{nk}(x)$ has the same sign as $a_{kn}(x)$, $m, k=1, \cdots, n-1$, $m\neq k$.

THEOREM 1.[1] *Let the $a_{ij}(x)$ be continuous and satisfy* (A), (B) *for some α_1, α_2 of $[a, b]$. Then there exists a unique solution of* (1) *satisfying the conditions*

$$(2) \qquad y_k(\alpha_1) = \beta_k, \qquad y_n(\alpha_2) = \beta_n, \qquad k = 1, \cdots, n-1,$$

where $\beta_1, \cdots, \beta_n$ are arbitrary real numbers.

PROOF. Let $(y_{i1}(x), \cdots, y_{in}(x))$, $i=1, \cdots, n$, be solutions of the homogeneous system

$$(3) \qquad y_i' = \sum_{j=1}^{n} a_{ij}(x)y_j$$

satisfying

Received by the editors April 15, 1960.

[1] A dual theorem may be given for $\alpha_2<\alpha_1$ if (B) is altered so that, if $a_{mn}(x)>0$, $a_{mk}(x)$ is required to have the opposite sign to $a_{kn}(x)$ and, if $a_{mn}(x)<0$, $a_{mk}(x)$ is required to have the same sign as $a_{kn}(x)$.

$$y_{ij}(\alpha_1) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad i, j = 1, \cdots, n. \tag{4}$$

Then the general solution $(y_1(x), \cdots, y_n(x))$ of (1) is given by

$$y_i(x) = c_1 y_{1i}(x) + c_2 y_{2i}(x) + \cdots + c_n y_{ni}(x) + y_{pi}(x),$$

where $(y_{pi}(x), \cdots, y_{pn}(x))$ is a particular solution of (1). Imposing the boundary conditions (2) to this system and simplifying according to (4), we obtain

$$\begin{aligned} c_1 &= \beta_1 - y_{p1}(\alpha_1) \\ &\vdots \\ c_{n-1} &= \beta_{n-1} - y_{p\,n-1}(\alpha_1) \\ c_1 y_{1n}(\alpha_2) + c_2 y_{2n}(\alpha_2) + \cdots + c_n y_{nn}(\alpha_2) &= \beta_n - y_{pn}(\alpha_2). \end{aligned}$$

This system has a solution, hence (2) can be satisfied uniquely, if $y_{nn}(\alpha_2) \neq 0$.

Assume at least one of $y_{n1}(x), \cdots, y_{nn}(x)$ has a zero on $(\alpha_1, \alpha_2]$. Of the $a_{mn}(x)$, $m = 1, \cdots, n-1$, let $a_{\nu_p n}(x)$, $p = 1, \cdots, r$, be those, if any, which are positive and let $a_{\mu_q n}(x)$, $q = 1, \cdots, s$, be those, if any, which are negative. Then, by (4), $y_{nn}(\alpha_1) = 1$, $y'_{n\nu_p}(\alpha_1) > 0$, $y'_{n\mu_q}(\alpha_1) < 0$. Hence $y_{nn}(x)$, $y_{n\nu_p}(x) > 0$, $y_{n\mu_q}(x) < 0$ to the immediate right of $x = \alpha_1$. Now since the $y_{nj}(x)$ are continuous it is possible to let c be the smallest zero of any of these functions on $(\alpha_1, \alpha_2]$. If $y_{n\nu_e}(c) = 0$, $\nu_e = n$ or $1 \leqq e \leqq r$, then $y'_{n\nu_e}(c) \leqq 0$. But, under this assumption,

$$\begin{aligned} y_{n\nu_e}(c) = {} & a_{\nu_e\nu_1}(c) y_{n\nu_1}(c) + \cdots + a_{\nu_e\nu_{e-1}}(c) y_{n\nu_{e-1}}(c) \\ & + a_{\nu_e\nu_{e+1}}(c) y_{n\nu_{e+1}}(c) + \cdots + a_{\nu_e\nu_r}(c) y_{n\nu_r}(c) \\ & + a_{\nu_e\mu_1}(c) y_{n\mu_1}(c) + \cdots + a_{\nu_e\mu_s}(c) y_{n\mu_s}(c) \\ & + a_{\nu_e n}(c) y_{nn}(c) \end{aligned}$$

is positive since, by the hypotheses and the above determined properties of $y_{nj}(x)$,

$$a_{\nu_e\nu_p}(c) y_{n\nu_p}(c), \ a_{\nu_e\mu_q}(c) y_{n\mu_q}(c), \ a_{\nu_e n}(c) y_{nn}(c) \geqq 0,$$

and since $y_{n1}(x), \cdots, y_{n\nu_{e-1}}(x), y_{n\nu_{e+1}}(x), \cdots, y_{nn}(x)$ cannot all vanish at $x = c$. If $y_{n\mu_f}(c) = 0$, $1 \leqq f \leqq s$, then $y'_{n\mu_f}(c) \geqq 0$. But, under this assumption, $y'_{n\mu_f}(c) < 0$ since

$$a_{\mu_f\mu_q}(c) y_{n\mu_q}(c), \ a_{\mu_f\nu_p}(c) y_{n\nu_p}(c), \ a_{\mu_f n}(c) y_{nn}(c) \leqq 0$$

and the functions $y_{n1}(x), \cdots, y_{n\mu_{f-1}}(x), y_{n\mu_{f+1}}(x), \cdots, y_{nn}(x)$ cannot all vanish at $x = c$. We now have a contradiction on the choice of c.

Hence none of $y_{n1}(x), \cdots, y_{nn}(x)$ vanishes on $(\alpha_1, \alpha_2]$ and the theorem follows.

COROLLARY. *Let the* $a_{ij}(x)$ *satisfy* (A), (B) *over* $[a, b]$. *Then Theorem* 1 *is valid without restricting the boundary points* α_1, α_2 *further than requiring them to belong to* $[a, b]$.

Conditions are not imposed on $a_{ii}(x)$, $i=1, \cdots, n$. If $a_{\gamma_t\gamma_t}(x)\equiv 0$ over (α_1, α_2) for $t=1, \cdots, u$ $(1\leqq u\leqq n)$, the same results can be obtained with weaker restrictions than (A), (B). Of the $a_{\gamma_t i}(x)$, $i=1, \cdots, n$, included in (B), we need require the positive functions only to be nonnegative and the negative functions to be only nonpositive with $a_{\gamma_t n}(\alpha_2)\neq 0$.

3. Three-point boundary conditions. For points

$$\alpha_1, \alpha_2, \alpha_3(\alpha_1 \leqq \alpha_2 \leqq \alpha_3)$$

of $[a, b]$ we define the following conditions:

C. $a_{mm}(x) = 0$ on (α_1, α_3); $a_{1n}(x), a_{n1}(x) > 0$ on $[\alpha_1, \alpha_3]$

$$m = 2, \cdots, n-1.$$

D. For each m, $2\leqq m\leqq n-1$, either

$$(1) \qquad \begin{aligned} a_{m1}(x) &\geqq 0 && \text{on } (\alpha_1, \alpha_3), \\ a_{m1}(\alpha_2) &> 0 && \text{or} \end{aligned}$$

$$(2) \qquad a_{m1}(x) \begin{cases} \leqq 0 & x \in (\alpha_1, \alpha_2], \\ \geqq 0 & x \in [\alpha_2, \alpha_3). \end{cases}$$

If (1) holds then

$$a_{1m}(x) > 0 \qquad \text{on } [\alpha_1, \alpha_3],$$

$$a_{nm}(x) \begin{cases} < 0 & x \in [\alpha_1, \alpha_2), \\ = 0 & x = \alpha_2, \\ > 0 & x \in (\alpha_2, \alpha_3], \end{cases}$$

$$a_{mn}(x) \begin{cases} \leqq 0 & x \in (\alpha_1, \alpha_2], \\ \geqq 0 & x \in [\alpha_2, \alpha_3). \end{cases}$$

If (2) holds then

$$a_{1m}(x) \begin{cases} < 0 & x \in [\alpha_1, \alpha_2), \\ = 0 & x = \alpha_2, \\ > 0 & x \in (\alpha_2, \alpha_3], \end{cases}$$

$$a_{nm}(x) > 0 \qquad \text{on } [\alpha_1, \alpha_3],$$
$$a_{mn}(x) \geqq 0 \qquad \text{on } (\alpha_1, \alpha_3),\ a_{mn}(\alpha_2) > 0.$$

E. For the case D(1):

$$a_{mk}(x) \begin{cases} \leqq 0 & x \in (\alpha_1, \alpha_2], \\ \geqq 0 & x \in [\alpha_2, \alpha_3) \end{cases} \qquad m > k,$$
$$a_{mk}(x) \equiv 0 \qquad \text{on } (\alpha_1, \alpha_3) \quad m < k \text{ if } a_{k1}(\alpha_2) > 0,$$
$$a_{mk} \geqq 0 \qquad \text{on } (\alpha_1, \alpha_3) \qquad \text{if } a_{k1}(\alpha_2) = 0.$$

For the case D(2):

$$a_{mk}(x) \geqq 0 \qquad \text{on } (\alpha_1, \alpha_3) \qquad \text{if } a_{k1}(\alpha_2) > 0,$$
$$a_{mk}(x) \begin{cases} \leqq 0 & x \in (\alpha_1, \alpha_2], \\ \geqq 0 & x \in [\alpha_2, \alpha_3) \end{cases} \qquad m > k,$$
$$a_{mk}(x) \equiv 0 \qquad \text{on } (\alpha_1, \alpha_3), \quad m < k \text{ if } a_{k1}(\alpha_2) = 0.$$
$$k = 2, \cdots, n-1,\ k \neq m.$$

F. For each m, $2 \leqq m \leqq n-1$, there exists a neighborhood, $(\alpha_2 - \delta_m, \alpha_2 + \delta_m)$, of α_2 for which $a_{m1}(x), \cdots, a_{mn}(x)$ do not all have a common zero.

LEMMA 1. *Let the $a_{ij}(x)$ be continuous and satisfy* (C), (D), (E), (F) *for some $\alpha_1, \alpha_2, \alpha_3$ of $[a, b]$ and let $(y_{11}(x), \cdots, y_{1n}(x)$ be the solution of* (3) *satisfying*

$$y_{1j}(\alpha_2) = \delta_{1j}, \qquad j = 1, \cdots, n. \tag{5}$$

Then $x = \alpha_2$ is an isolated zero of $y_{1f}(x), f = 2, \cdots, n$.

PROOF. Of the $a_{f1}(x), f = 2, \cdots, n$, let $a_{\nu_p 1}(x)$, $\nu_1 < \nu_2 < \cdots < \nu_r$ be those which are positive at α_2 and let $a_{\mu_q 1}(x)$, $\mu_1 < \mu_2 < \cdots < \mu_s$, be those, if any, which are zero at α_2. Then, by (5), $y_{11}(\alpha_2) = 1$, $y'_{1\nu_p}(\alpha_2) > 0$, $p = 1, \cdots, r$. Since $y_{1\nu_p}(x)$ is continuous and vanishes at $x = \alpha_2$, there exists a $\delta > 0$ such that

$$y_{11}(x), y_{1\nu_p}(x) < 0 \qquad \text{on } (\alpha_2 - \delta, \alpha_2),$$
$$y_{11}(x), y_{1\nu_p}(x) > 0 \qquad \text{on } (\alpha_2, \alpha_2 + \delta).$$

With this and the hypotheses, we have that

$$y'_{1\mu_1}(x) = a_{\mu_1 1}(x) y_{11}(x) + a_{\mu_1 \nu_1}(x) y_{1\nu_1}(x) + \cdots + a_{\mu_1 \nu_r}(x) y_{1\nu_r}(x)$$

is negative over $(\alpha_2 - \delta_1, \alpha_2)$ and positive over $(\alpha_2, \alpha_2 + \delta_1)$, where $\delta_1 = \min(\delta, \delta_{\mu_1})$. Hence, since $y_{1\mu_1}(x)$ vanishes at $x = \alpha_2$ and is continuous, $y_{1\mu_1}(x) > 0$ on $(\alpha_2 - \delta_1, \alpha_2)$, $(\alpha_2, \alpha_2 + \delta_1)$. We now have that

$$y'_{1\mu_2}(x) = a_{\mu_2 1}(x)y_{11}(x) + a_{\mu_2\mu_1}(x)y_{1\mu_1}(x) + a_{\mu_2\nu_1}(x)y_{1\nu_1}(x) + \cdots + a_{\mu_2\nu_r}(x)y_{1\nu_r}(x)$$

is negative over $(\alpha_2-\delta_2, \alpha_2)$ and positive over $(\alpha_2, \alpha_2+\delta_2)$, where $\delta_2 = \min(\delta_1, \delta_{\mu_2})$. This implies that $y_{1\mu_2}(x) > 0$ on $(\alpha_2 - \delta_2, \alpha_2)$, $(\alpha_2, \alpha_2+\delta_2)$. By continuing in this way we find $y_{1\mu_q}(x) > 0$ in a neighborhood of $x=\alpha_2$ for $q=1, \cdots, s$.

We have shown that each $y_{1f}(x)$ is either positive or negative to the immediate left of $x=\alpha_2$ and positive to the immediate right of $x=\alpha_2$. Hence α_2 is an isolated zero of $y_{1f}(x), f=2, \cdots, n$.

LEMMA 2. *Let the $a_{ij}(x)$ be continuous and satisfy* (C), (D), (E), (F) *for some $\alpha_1, \alpha_2, \alpha_3$ of $[a, b]$ and let $(y_{n1}(x), \cdots, y_{nn}(x))$ be the solution of* (3) *satisfying*

$$y_{nj}(\alpha_2) = \delta_{nj}, \qquad j = 1, \cdots, n. \tag{6}$$

Then $x=\alpha_2$ is an isolated zero of $y_{ne}(x)$, $e=1, \cdots, n-1$.

PROOF. By a process similar to that in the proof of Lemma 1, we find that $y_{n\nu_p}(x)$, $p=1, \cdots, r$, is positive in a neighborhood of $x=\alpha_2$ and $y_{n\mu_q}(x)$, $q=1, \cdots, s$, is negative to the immediate left of $x=\alpha_2$, positive to the immediate right of $x=\alpha_2$.

THEOREM 2. *Let the $a_{ij}(x)$ be continuous and satisfy* (C), (D), (E), (F) *for some $\alpha_1, \alpha_2, \alpha_3$ of $[a, b]$. Then there exists a unique solution of* (1) *satisfying*

$$y_1(\alpha_1) = \beta_1,\ y_m(\alpha_2) = \beta_m,\ y_n(\alpha_3) = \beta_n, \qquad m = 2, \cdots, n-1, \tag{7}$$

where $\beta_1, \cdots, \beta_n$ are arbitrary real numbers.

PROOF. Let $(y_{i1}(x), \cdots, y_{in}(x))$, $i=1, \cdots, n$, be solutions of (3) satisfying

$$y_{ij}(\alpha_2) = \delta_{ij}, \qquad i, j = 1, \cdots, n. \tag{8}$$

Then the general solution $(y_1(x), \cdots, y_n(x))$ of (1) is given by

$$y_i(x) = c_1y_{1i}(x) + c_2y_{2i}(x) + \cdots + c_ny_{ni}(x) + y_{pi}(x),$$

where $(y_{p1}(x), \cdots, y_{pn}(x))$ is a particular solution (1). Imposing the boundary conditions (7) and simplifying according to (8), we obtain

$$\begin{aligned} c_1y_{11}(\alpha_1) + c_2y_{21}(\alpha_1) + \cdots + c_ny_{n1}(\alpha_1) &= \beta_1 - y_{p1}(\alpha_1), \\ c_2 &= \beta_2 - y_{p2}(\alpha_2), \\ &\vdots \\ c_{n-1} &= \beta_{n-1} - y_{p\,n-1}(\alpha_2), \\ c_1y_{1n}(\alpha_3) + c_2y_{2n}(\alpha_3) + \cdots + c_ny_{nn}(\alpha_3) &= \beta_n - y_{pn}(\alpha_3). \end{aligned}$$

This system has a solution, hence (7) can be satisfied uniquely, if $y_{11}(\alpha_1)y_{nn}(\alpha_3)-y_{1n}(\alpha_3)y_{n1}(\alpha_1)$ is nonzero. We proceed to show this is the case.

Let $a_{\nu_p 1}(x)$, $a_{\mu_p 1}(x)$ be defined as above. We first prove

(a) $$y_{11}(x) > 0 \qquad \text{on } [\alpha_1, \alpha_2).$$

Assume at least one of the functions $y_{11}(x), \cdots, y_{1n}(x)$ has a zero on $[\alpha_1, \alpha_2)$. By virtue of Lemma 1 and the continuity of $y_{1j}(x)$, $j=1, \cdots, n$, it is possible to let c be the largest zero of any of these functions on $[\alpha_1, \alpha_2)$. Hence the sign of $y_{1j}(x)$ as found in Lemma 1 holds over the interval (c, α_2).

If $y_{11}(c)=0$, then, since $y_{11}(x)>0$ on (c, α_2), $y_{11}'(c)\geqq 0$. But, under this assumption,

$$\begin{aligned} y_{11}'(c) = {} & a_{1\nu_1}(c)y_{1\nu_1}(c) + \cdots + a_{1\nu_r}(c)y_{1\nu_r}(c) \\ & + a_{1\mu_1}(c)y_{1\mu_1}(c) + \cdots + a_{1\mu_s}(c)y_{1\mu_s}(c) \end{aligned}$$

is negative since $a_{1\nu_p}(c)y_{1\nu_p}(c)$, $a_{1\mu_q}(c)y_{1\mu_q}(c)\leqq 0$ and the functions $y_{12}(x), \cdots, y_{1n}(x)$ cannot all vanish at $x=c$. Hence $y_{11}(c)\neq 0$. For a similar reason, $y_{1n}(c)\neq 0$. The function $y_{1\mu_q}(x)$, $q=1, \cdots, s$, does not vanish at $x=c$ since, from the proof of Lemma 1, its derivative does not change sign over (c, α_2). For any e, $1\leqq e\leqq r-1$,

$$\begin{aligned} y_{1\nu_e}'(x) = {} & a_{\nu_e 1}(x)y_{11}(x) + a_{\nu_e \mu 1}(x)y_{1\mu_1}(x) + \cdots \\ & + a_{\nu_e \mu_s}(x)y_{1\mu_s}(x) + a_{\nu_e\nu_1}(x)y_{1\nu_1}(x) + \cdots \\ & + a_{\nu_e\nu_{e-1}}(x)y_{1\nu_{e-1}}(x) + a_{\nu_e\nu_r}(x)y_{1\nu_r}(x) \end{aligned}$$

is positive to the immediate left of $x=\alpha_2$ and nonnegative over (c, α_2). Thus since $y_{1\nu_e}(\alpha_2)=0$, $y_{1\nu_e}(x)$ cannot vanish at $x=c$. We now have a contradiction on the choice of c. Hence $y_{11}(x), \cdots, y_{1n}(x)$ do not vanish on $[\alpha_1, \alpha_2)$.

The following statements are proved in a similar manner to (*a*):

(b) $$y_{n1}(x) < 0 \qquad \text{on } [\alpha_1, \alpha_2).$$

(c) $$y_{1n}(x) > 0 \qquad \text{on } (\alpha_2, \alpha_3].$$

(d) $$y_{nn}(x) > 0 \qquad \text{on } (\alpha_2, \alpha_3].$$

We now have $y_{11}(\alpha_1)y_{nn}(\alpha_3)-y_{1n}(\alpha_3)y_{n1}(\alpha_1)>0$ for $\alpha_1<\alpha_2<\alpha_3$. If $\alpha_1=\alpha_2<\alpha_3$ ($\alpha_1<\alpha_2=\alpha_3$) then the determinant in question is $y_{nn}(\alpha_3)>0$ ($y_{11}(\alpha_1)>0$). Hence there exists a unique solution for the c_i. This in turn gives a unique solution of (1) satisfying (7).

COROLLARY. *Let the* $a_{ij}(x)$ *satisfy* (C), (D), (E), (F) *over the interval* $[a, \alpha_2)$, $(\alpha_2, b]$ *for some* $\alpha_2\in[a, b]$. *Then Theorem* 2 *is valid without* *restricting the boundary points* α_1, α_3 *further than requiring them to belong to* $[a, \alpha_2]$, $[\alpha_2, b]$, *respectively.*

Conditions are not imposed on $a_{11}(x)$, $a_{nn}(x)$. If these functions are identically zero over (α_1, α_3) Theorem 2 follows for weaker restrictions than (C), (D). For the $a_{1f}(x)$, $a_{ne}(x)$, $f=2, \cdots, n$, $e=1, \cdots, n-1$, it is sufficient to require that the positive functions be nonnegative, the negative functions be nonpositive and $a_{1n}(\alpha_2)$, $a_{n1}(\alpha_2)>0$.

AUBURN UNIVERSITY

A MOORE SPACE ON WHICH EVERY REAL-VALUED CONTINUOUS FUNCTION IS CONSTANT

STEVE ARMENTROUT

F. B. Jones [2] recently gave an example of a Moore space Λ_∞ in which there exists a point x such that Λ_∞ is not completely regular at x. It is easy to modify the construction used by Jones so as to obtain a Moore space A in which there exist distinct points a and b such that for every real-valued continuous function f on A, $f(a)=f(b)$. Upon applying Urysohn's process of condensation of the singularities of the space A [4], in a manner similar to that used by Hewitt [1], there results a Moore space X on which every real-valued continuous function is constant.

Throughout this paper, J denotes the set of positive integers. A sequence is a function on J, and if f is a sequence and $n\in J$, then f_n denotes $f(n)$.

By a Moore space is meant a topological space X whose topology has a basis consisting of sets termed *regions*, satisfying the following condition (axiom 1_3, that is, parts 1, 2, and 3 of axiom 1, of [3]): There exists a sequence G such that (1) if $n\in J$, G_n is a collection of regions covering X, (2) if $n\in J$, $G_{n+1}\subset G_n$, and (3) if r is a region, $x\in r$, and $y\in r$, then there exists a positive integer n such that if $g\in G_n$ and $x\in g$, then $\bar{g}\subset(r-\{x\})\cup\{y\}$. The following characterization of a Moore space will be used in this paper: X is a Moore space if and only if X is a regular Hausdorff space for which there exists a sequence G of open coverings of X such that if U is an open set and

---

Presented to the Society, September 3, 1959; received by the editors February 2, 1960.