SUMMABILITY-PRESERVING FUNCTIONS

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The questions answered in this paper suggest themselves naturally. The first lemma is a consequence of a result of R. C. Buck.

LEMMA. *Let f be a function of a complex variable and A a Toeplitz matrix such that $\{f(x_n)\}$ is A-summable whenever $\{x_n\}$ converges. Then f is continuous.*

PROOF. Every subsequence of $\{x_n\}$ converges, so that every subsequence of $\{f(x_n)\}$ is A-summable, by hypothesis. By [1], applied to complex sequences, $\{f(x_n)\}$ is actually convergent. (For the proof of [1] can be modified to apply to complex sequences. Or, see [2, Theorem 2], for a proof in a slightly more general context.) Let $\{x_n\}$ converge to x. The sequence $x_1, x, x_2, x, \cdots$ is also convergent, so that the sequence $f(x_1), f(x), f(x_2), f(x), \cdots$ converges. This shows that $\{f(x_n)\}$ converges to $f(x)$ whenever $\{x_n\}$ converges to x. Thus the continuity of f is proved.

The converse of the lemma is a consequence of the definition of Toeplitz matrix. The lemma is of course also true for real-valued functions of a real variable. The same comment applies also after the following theorem.

THEOREM. *Let $A=(a_{in})$, $1\leq i,n<\infty$, be a Toeplitz matrix, and f a function of a complex variable, such that if $\{x_n\}$ is a bounded $(C, 1)$-summable sequence, then $\{f(x_n)\}$ is A-summable. Then f is linear. If in addition f is not the constant function, then A sums every bounded $(C, 1)$-summable sequence to its $(C, 1)$ sum.*

PROOF. Every convergent sequence is a bounded $(C, 1)$-summable sequence. Thus f satisfies the hypotheses of the lemma. Hence f is continuous. We will prove that $f((a+b)/2)=(f(a)+f(b))/2$ for all a and b. It is well known that a continuous function with this property is linear.

Consider the sequence $a, b, a, b, \cdots$, which is a bounded $(C, 1)$-summable sequence. In fact, we may interpolate any number of terms $(a+b)/2$ between successive terms of this sequence, infinitely many if we wish, and still have a bounded $(C, 1)$-summable sequence. To prove this, let us examine the average of the first n terms of such an interpolated sequence. These first n terms consist of rb's, r or $r+1$ a's,

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and $n-2r$ or $n-2r-1((a+b)/2)$'s, say. The average of these n terms is therefore equal to $(1/n)(r(a+b)+(n-2r)(a+b)/2)$ or $(1/n)(r(a+b)+a+(n-2r-1)(a+b)/2)$, that is, to $(a+b)/2$ or $((a+b)/2)(1-1/n)+a/n$. As n approaches infinity, the Cesaro means of such an interpolated sequence therefore converge to $(a+b)/2$. This proves the assertion. By the hypothesis of this theorem, we conclude that any sequence $f(a), f(b), f(a), f(b), \cdots$, with any number of $f((a+b)/2)$'s interpolated into it is A-summable. We shall use this fact shortly.

If f is the constant function, there is nothing more to prove. Throughout the remainder of the proof, f will be nonconstant. Let a, b be such that $f(a)=p$, $f(b)=q$, with $p\neq q$. The sequence $p, q, p, q, \cdots$ is A-summable by hypothesis. Then the sequence $p-q, 0, p-q, 0, \cdots$ is also A-summable. For this sequence is obtained from the preceding sequence by subtracting the sequence $q, q, q, \cdots$. Upon dividing each term of $p-q, 0, p-q, 0, \cdots$ by $p-q$, we conclude that the sequence $1, 0, 1, 0, \cdots$ is A-summable. In other words, $\lim_{i\to\infty}\sum_{n \text{ odd}} a_{in}$ exists; call it r. Since for every Toeplitz matrix, $\lim_{i\to\infty}\sum_{n=1}^{\infty} a_{in}=1$, we must have $\lim_{i\to\infty}\sum_{n \text{ even}} a_{in}=1-r$.

Now consider any sequence $f(a), f(b), f(a), f(b), \cdots$. We shall interpolate $f((a+b)/2)$'s into this sequence in such a way that we obtain a new sequence, which we know to be A-summable by the above argument, and yet which has a subsequence of its sequence of auxiliary means under A convergent to $f((a+b)/2)$, and another subsequence of its sequence of auxiliary means under A convergent to $rf(a)+(1-r)f(b)$. But since the sequence of auxiliary means under A of the interpolated sequence converges, and since a convergent sequence has but one limit point, we must conclude that these two limit points coincide.

Let M be the maximum of the absolute values of the three numbers $f(a), f(b), f((a+b)/2)$. Let N_1 be even and so large that $\sum_{n>N_1}|a_{1n}| <1/2M$. Then for any sequence $\{c_n\}$ composed only of terms chosen from $f(a), f(b), f((a+b)/2)$, whose first N_1 terms are the first N_1 terms of the sequence $f(a), f(b), f(a), f(b), \cdots$, we observe that $\sum_{n=1}^{\infty} a_{in}c_n$ differs in absolute value from $f(a)\sum_{n \text{ odd}} a_{1n}+f(b)\sum_{n \text{ even}} a_{1n}$ by less than $2M\cdot 1/2M=1$. Let $i_1=1$ and choose $i_2>i_1$ and so large that $\sum_{n=1}^{N_1}|a_{i_2,n}|<1/2M$. We now start interpolating terms $f((a+b)/2)$. Let N_2 be even, $>N_1$, and so large that $\sum_{n>N_2}|a_{i_2,n}|<1/4M$. Then any sequence $\{c_n\}$ composed only of terms $f(a)$, $f(b)$, $f((a+b)/2)$, whose terms from N_1+1 up to N_2 are all equal to $f((a+b)/2)$, has the property that $\sum_{n=1}^{\infty} a_{i_2,n}c_n$ differs in absolute value from $f((a+b)/2)\sum_{n=1}^{\infty} a_{i_2,n}$ by less than $M\cdot 1/2M+2M\cdot 1/4M=1$. We shall now find $i_3 > i_2$ and so large that $\sum_{n=1}^{N_2} |a_{i_3,n}| < 1/4M$. Now choose N_3 even, $> N_2$, and so large that $\sum_{n>N_3} |a_{i_3,n}| < 1/8M$. We now leave in our sequence $\{c_n\}$ that we are constructing, terms a, b, a, b, $\cdots$ starting from N_2+1 and stopping at N_3. Any sequence $\{c_n\}$ with such terms in the indicated positions and its remaining terms chosen from among $f(a)$, $f(b)$, $f((a+b)/2)$, has the property that $\sum_{n=1}^{\infty} a_{i_3,n}c_n$ differs in absolute value from $f(a)\sum_{n \text{ odd}} a_{i_3,n} + f(b)\sum_{n \text{ even}} a_{i_3,n}$ by less than $M \cdot 1/4M + 2M \cdot 1/8M = 1/2$. Choose $i_4 > i_3$ and so large that $\sum_{n=1}^{N_3} |a_{i_4,n}| < 1/4M$. Then choose N_4 even, $> N_3$, and so large that $\sum_{n>N_4} |a_{i_4,n}| < 1/8M$. Now interpolate $f((a+b)/2)$ from the N_3+1 to the N_4 position. Any sequence $\{c_n\}$ with $f((a+b)/2)$ in these positions and its remaining terms chosen from among $f(a)$, $f(b)$, $f((a+b)/2)$, has the property that $\sum_{n=1}^{\infty} a_{i_4,n}c_n$ differs in absolute value from $f((a+b)/2)\sum_{n=1}^{\infty} a_{i_4,n}$ by less than $M \cdot 1/4M + 2M \cdot 1/8M = 1/2$. Continuing in this fashion, we finally obtain a sequence $\{c_n\}$ with the property that

$$\left| \sum_{n=1}^{\infty} a_{i_{2k-1},n}c_n - f(a) \sum_{n \text{ odd}} a_{i_{2k-1},n} - f(b) \sum_{n \text{ even}} a_{i_{2k-1},n} \right| < \frac{1}{k}, \qquad k = 1, 2, \cdots,$$

and

$$\left| \sum_{n=1}^{\infty} a_{i_{2k},n}c_n - f\left(\frac{a+b}{2}\right) \sum_{n=1}^{\infty} a_{i_{2k},n} \right| < \frac{1}{k}, \qquad k = 1, 2, \cdots.$$

Given any $\epsilon > 0$, if $k > 2/\epsilon$ and also so large that

$$\left| \sum_{n \text{ odd}} a_{i_{2k-1},n} - r \right| < \frac{\epsilon}{4M}, \quad \left| \sum_{n \text{ even}} a_{i_{2k-1},n} - (1-r) \right| < \frac{\epsilon}{4M},$$

$$\left| \sum_{n=1}^{\infty} a_{i_{2k},n} - 1 \right| < \frac{\epsilon}{2M},$$

then we have

$$\left| \sum_{n=1}^{\infty} a_{i_{2k-1},n}c_n - rf(a) - (1-r)f(b) \right| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon,$$

$$\left| \sum_{n=1}^{\infty} a_{i_{2k},n}c_n - f\left(\frac{a+b}{2}\right) \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

From this, we conclude that the subsequence of the auxiliary means of $\{c_n\}$ corresponding to those rows of A indexed by i_{2k-1}, $k = 1, 2, \cdots$, converges to $rf(a) + (1-r)f(b)$. Similarly, the subsequence of the auxiliary means of $\{c_n\}$ under A corresponding to those rows of A

indexed by i_{2k}, $k=1, 2, \cdots$, converges to $f((a+b)/2)$. Since a convergent sequence has but one limit point, we conclude that $f((a+b)/2) = rf(a)+(1-r)f(b)$. Starting with the sequence $b, a, b, a, \cdots$, we likewise conclude that $f((b+a)/2)=rf(b)+(1-r)f(a)$. Thus for all a and b, $rf(a)+(1-r)f(b)=rf(b)+(1-r)f(a)$. Since f is nonconstant, we can choose a, b such that $f(a) \neq f(b)$. Then $r(f(a)-f(b)) = (1-r)(f(a)-f(b))$. Thus $r=1-r$, or $r=1/2$. Then $f((a+b)/2) = (1/2)f(a)+(1-1/2)f(b)$, that is, $f((a+b)/2)=(f(a)+f(b))/2$ for all a and b. Since f is continuous, we conclude that f is linear.

To prove the last part of the theorem, let $f(z)=cz+d$ with $c\neq 0$. The hypothesis of the theorem tells us that $\{cx_n+d\}$ is A-summable whenever $\{x_n\}$ is a bounded $(C, 1)$-summable sequence. Subtracting the sequence $d, d, d, \cdots$ from this sequence and dividing by c, we find that $\{x_n\}$ is A-summable whenever $\{x_n\}$ is a bounded $(C, 1)$-summable sequence. Theorem 1 of [3] is now exactly what one needs to conclude that A sums every bounded $(C, 1)$-summable sequence to its $(C, 1)$ sum. This concludes the proof of the theorem.

The theorem is of course false for summability methods (as opposed to $(C, 1)$) whose convergence field is too small.

A question which arises in connection with the theorem has been answered by Professor H. Hanani: Namely, if the Toeplitz matrix A sums every bounded $(C, 1)$-summable sequence, does it sum every $(C, 1)$-summable sequence? The answer is "no." For let $a_{in}=1/i$, $1\leqq n\leqq i$, $a_{i,i^2}=1/i$, $a_{in}=0$ otherwise. The matrix $A=(a_{in})$ is a Toeplitz matrix which sums every bounded $(C, 1)$-summable sequence, as is easy to see. But the sequence $\{(-1)^{n+1}n^{1/2}\}, n=1, 2, \cdots$, is $(C, 1)$-summable (to zero), whereas the sequence of its odd auxiliary means under A converges to $+1$, its even ones to -1. Another question whose answer is "no" is this: if the Toeplitz matrix A sums every bounded $(C, 1)$-summable sequence, does it give the $(C, 1)$ sum for any $(C, 1)$-summable sequence which it happens to sum? For let $A=(a_{in})$ with $a_{in}=1/i$, $1\leqq n\leqq i$, $a_{i,i^2}=(-1)^{i+1}\cdot 1/i$, $a_{in}=0$ otherwise. Then the same sequence as used above is summable by this matrix to 1, not zero.

Bibliography

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