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QUOTIENT GROUPS OF REDUCED ABELIAN GROUPS

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Let G be a reduced torsion p-group. (In this paper, group will mean Abelian group.) Let B be a basic subgroup of G. It is well known that $|B|^{\aleph_0} \geqq |G|$, where $|S|$ denotes the cardinal of the set S. Fuchs gives a proof of this in [1, p. 102], and attributes it to Kulikov. This has turned out to be a very useful fact, and the purpose of this short note is to generalize it. Now, as is generally known, every torsion group G has a basic subgroup B; that is, a pure subgroup B that is a direct sum of cyclic groups, and such that G/B is divisible. To obtain such a B, simply take B_p to be a basic subgroup of the p-component of G, and let $B = \sum_p \oplus B_p$. It is easy to see that in this more general situation $|B|^{\aleph_0} \geqq |G|$ still holds, and in fact follows from the corresponding statement for p-groups. The generalization we will prove is the following

THEOREM. *Let G be a reduced torsion group, and let H be a subgroup of G such that G/H is divisible. Then* $|H|^{\aleph_0} \geqq |G|$.

PROOF. Our proof uses some homological results of Harrison in [2]. Notice that we may assume that $|H| < |G|$, that G is infinite, and hence that $|G/H| = |G|$. Let Q and Z denote the additive group of rationals and integers, respectively. From the exact sequence

$$0 \to H \to G \to G/H \to 0,$$

we get the exact sequence

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$$0 \to \mathrm{Hom}(Q/Z, G/H) \to \mathrm{Ext}(Q/Z, H) \to \mathrm{Ext}(Q/Z, G) \to 0,$$

since H and G are reduced, and Q/Z and G/H are divisible. If H is of bounded order, then $\mathrm{Ext}(Q/Z, H) \cong H$. (See [2].) However, $\mathrm{Hom}(Q/Z, G/H)$ is torsion-free [2, p. 371], so that H must be of unbounded order, and in particular $|H| \geqq \aleph_0$. Therefore we may assume $|G| > 2^{\aleph_0}$. Hence G/H is the direct sum of $|G|$ copies of $Z(p^\infty)$'s, for various primes p. Now for a particular $Z(p^\infty)$, $|\mathrm{Hom}(Q/Z, Z(p^\infty))| = 2^{\aleph_0}$, being the p-adic integers. Thus $|\mathrm{Hom}(Q/Z, G/H)| \geqq 2^{\aleph_0}|G| = |G|$, and from the exact sequence above we see that $\mathrm{Ext}(Q/Z, H)$ has a subgroup of cardinal $|G|$. But $\mathrm{Ext}(Q/Z, H)$ is a quotient group of a group of functions of two variables of Q/Z into H (see [2, p. 368]) and hence $|\mathrm{Ext}(Q/Z, H)| \leqq |H|^{\aleph_0}$. Therefore $|H|^{\aleph_0} \geqq |G|$.

We conclude with the following remarks. First, if H is a basic subgroup, we have obviously shown $|H|^{\aleph_0} \geqq |G|$. Secondly, suppose H is of bounded order. Write $G/H = D \oplus R$, where D is divisible and R is reduced. $\mathrm{Hom}(Q/Z, G/H) = \mathrm{Hom}(Q/Z, D \oplus R) \cong \mathrm{Hom}(Q/Z, D) \oplus \mathrm{Hom}(Q/Z, R)$. $\mathrm{Hom}(Q/Z, D)$ is torsion-free and is isomorphic to a subgroup of $\mathrm{Ext}(Q/Z, H) \cong H$. Hence $D = 0$ since D is torsion, and we see that *if H is a subgroup of G of bounded order, then G/H is reduced.* Finally, we easily see from the proof of our theorem above that *if $G/H = D \oplus R$, with D divisible, R reduced, and $|D| = |G|$, then $|H|^{\aleph_0} \geqq |G|$.*

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