RELATIONS BETWEEN FUNCTION SPACES

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1. Introduction. Let B be a conditionally σ-complete ring of subsets of a set S. We assume that $W(e)$ is a positive function defined for $e\in B$ with the properties:

(1) W is increasing: $W(e_1)\leqq W(e_2)$ if $e_1\subset e_2$,

(2) W is concave: $W(e_1\cup e_2)+W(e_1\cap e_2)\leqq W(e_1)+W(e_2)$, $e_1, e_2\in B$,

(3) W is continuous: $W(e_n)\to 0$ if $e_n\in B$ decreases to 0.

If f is a B-measurable real function defined on S, then for each $y>0$ the set $f^{-1}(y)=[x: |f(x)|>y]$ belongs to B. The space $\Lambda(W)$ [**3; 1**] consists of all measurable functions f for which

$$\|f\| = \int_0^{+\infty} W(f^{-1}(y))dy < +\infty. \tag{4}$$

With this norm, $\Lambda(W)$ is a Banach space and even a Banach lattice, i.e., a vector lattice with the property that $|g|\leqq f$, $f\in\Lambda(W)$ for a measurable g implies $g\in\Lambda(W)$ and $\|g\|\leqq\|f\|$. Condition (3) ensures [**1**, Theorem 5] that the norm (4) is continuous with respect to monotone limits.

One obtains examples of such functions W as follows. Let (S, B, μ) be a measure space with measure μ, which we shall in this paper assume nonatomic. If $\Phi_0(u)$ is a positive increasing concave function of $u\geqq 0$ with $\Phi_0(0+)=0$, then one sees that $W(e)=\Phi_0(\mu e)$ satisfies (1), (2), (3).

An Orlicz space $L_\Phi(S)$ is defined by an increasing convex function $\Phi(u)$, $u\geqq 0$ with $\Phi(0+)=0$ and $\Phi'(u)\to\infty$, and by a measure space (S, B, μ).

We have $f\in L_\Phi$ if

$$\|f\| = \sup \int_S |f|\, g d\mu < +\infty; \tag{5}$$

the supremum is taken for all measurable $g\geqq 0$ with $\int_S\Psi(g)d\mu\leqq 1$, and $\Psi(u)$ is the conjugate (in the sense of Young) of Φ. This means that $\Phi(u)$ and $\Psi(u)$ are integrals over $(0, u)$ of two monotone positive finite functions $\phi(t)$, $\psi(t)$ which increase to $+\infty$ for $t\to+\infty$ and are

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inverses of each other. For an exposition of the theory of Orlicz spaces see [5; 2].

The problems solved in this note are the following. When is a given space L_Φ also some space $\Lambda(W)$? Or when does a given $\Lambda(W)$ coincide with some L_Φ? Theorems 2 and 3 below show that this can happen only in exceptional cases. We conjecture that this is also true if the L_Φ are replaced by modulared semi-ordered spaces of Nakano [4].

2. **Largest space $\Lambda(W)$ contained in L_Φ.** We start by deriving consequences from $X \subset Y$, where X and Y are Banach lattices of B-measurable functions. Let X have the "Fatou property": a measurable function f belongs to X and $\|f\| = \lim \|f_n\|$ if $0 \leqq f_n(x) \uparrow f(x)$ and $\sup \|f_n\| < +\infty$. (Thus, spaces Λ [1, Theorem 5] and L_Φ have this property.) Then $X \subset Y$ *implies that for some constant* C,

$$(6) \qquad \|\chi_e\|_Y \leqq C\|\chi_e\|_X.$$

Here χ_e is the characteristic function of an arbitrary set $e \in B$. For otherwise $e_n \in B$ would exist with $\|\chi_{e_n}\|_Y > n^3\|\chi_{e_n}\|_X$, and for the function $f = \sum_1^\infty f_n$, $f_n = \chi_{e_n}\|\chi_{e_n}\|_X^{-1}$ we would have $f \in X$, $f \notin Y$.

From this and the definition (4) we obtain for two spaces Λ, corresponding to the same (S, B): *The inclusion* $\Lambda(W_1) \subset \Lambda(W_2)$ *is equivalent to* $W_2(e) \leqq CW_1(e)$, $e \in B$, *for some constant* C.

THEOREM 1. *Let* L_Φ *be an Orlicz space for the measure space* (S, B, μ). *Then there exists a largest space* $\Lambda(W)$ *of* B*-measurable functions contained in* L_Φ, *namely the space* $\Lambda(W_0)$ *with*

$$(7) \quad W_0(e) = \Phi_0(\mu e), \quad \Phi_0(x) = x\Psi^{-1}\left(\frac{1}{x}\right), \quad x > 0, \quad \Phi_0(0) = 0.$$

PROOF. Consider the function

$$(8) \qquad W_0(e) = \|\chi_e\|_{L_\Phi} = \sup_{\int\Psi(g)d\mu = 1;\, g \geqq 0} \int_e g(x)d\mu(x).$$

Assuming $\mu e > 0$, let g be an arbitrary positive measurable function with $\int_e \Psi(g)d\mu = 1$; we define C and C_1 by

$$\mu e\Psi(C) = 1 \quad \text{and} \quad C_1 = (\mu e)^{-1}\int_e g d\mu.$$

By Jensen's inequality

$$\mu e\Psi(C_1) \leqq \int_e \Psi(g)d\mu = 1,$$

hence $\Psi(C_1) \leqq \Psi(C)$ and $C_1 \leqq C$. This means that $\int_e C d\mu \geqq \int_e g d\mu$, i.e., that the supremum in (8) is attained for $g = C$. Thus we obtain that $W_0(e)$ defined by (8) is identical with the function (7). Next we have, if $x \neq 0$, $y = \Psi^{-1}(1/x)$, $t = \psi(y)$, and if $\Psi'(y)$ exists, i.e., the function ψ is continuous at y,

$$\begin{aligned} \Phi_0'(x) &= \Psi^{-1}\left(\frac{1}{x}\right) - \frac{1}{x\Psi'(y)} \\ &= y - \frac{\Psi(y)}{\Psi'(y)} \qquad (9) \\ &= \frac{\Phi(t)}{t}, \end{aligned}$$

because in this case $\Psi(y) + \Phi(t) = yt$.

If ψ is discontinuous at y, then still the right and the left derivatives of Φ_0 at x are equal to $t^{-1}\Phi(t)$ with $t = \psi(y-)$ or $t = \psi(y+)$, respectively. This shows that $\Phi_0(x)$ has positive decreasing derivatives, hence $\Phi_0(x)$ is an increasing concave function. Since $y^{-1}\Psi(y) \to \infty$ for $y \to \infty$, we have $\Phi_0(0+) = 0$. Thus the function (7) satisfies (1), (2), (3).

Considering $Y = \Lambda(W_0)$, we see that $\|f\|_Y \geqq \|f\|_{L_\Phi}$ holds [1, Theorems 1, 2] for each step function f. The same relation for an arbitrary $f \geqq 0$ in Y follows from the Fatou property of L_Φ: from (4) one derives that f is the limit of an increasing sequence of positive step functions f_n for which $\limsup \|f_n\|_{L_\Phi} \leqq \lim \|f_n\|_Y = \|f\|_Y$. Thus $f \in L_\Phi$ and $Y \subset L_\Phi$. Finally, any other Λ-space contained in L_Φ is also contained in $\Lambda(W_0)$. For $\Lambda(W) \subset L_\Phi$ implies by (6) $W_0(e) \leqq CW(e)$ and $\Lambda(W) \subset \Lambda(W_0)$.

3. **Conditions for the equality of the spaces** Λ, L_Φ. From (4) one derives that each function $f \in \Lambda(W)$ can be approximated in the $\Lambda(W)$-norm by step functions. The spaces L_Φ do not possess in general this property. Let $l = \mu S$; this approximation is possible if and only if constants M and $u_0 > 0$ exist with

$$\Phi(2u) \leqq M\Phi(u) \quad \text{for} \quad \begin{cases} u \geqq u_0 & \text{if } l < +\infty, \\ u > 0 & \text{if } l = +\infty. \end{cases} \qquad (10)$$

Sufficiency of this condition is known [5, p. 84]. We shall prove its necessity for $l < +\infty$ (the other case is treated in a similar fashion).

If (10) were not true, we could construct by induction sequences $u_n \to \infty$ and $\delta_n > 0$ with $\sum_{k=1}^{n} \delta_k < l$ as follows. At the nth step, we take $u_n > n$ so that $\Phi(2u_n) > 2^n \Phi(u_n)$. By the induction hypothesis, $\sum_1^{n-1} \delta_k$

$<l$. We can assume u_n so large that $\sum_1^n \delta_k < l$ with $\delta_n = 2^{-n}\Phi(u_n)^{-1}$. Since μ is nonatomic and therefore full-valued, we can find disjoint measurable sets $e_n \in B$ with $\mu e_n = \delta_n$. Put $f(x) = 3u_n$ for $x \in e_n$, $n = 1, 2, \cdots$, $f(x) = 0$ for all other points of S. Then $f \in L_\Phi$, since (see [5, p. 79])

$$\int_S \Phi\left(\frac{1}{3}f\right) d\mu = \sum_{n=1}^{\infty} \Phi(u_n)\delta_n < +\infty.$$

On the other hand, if f_1 takes only a finite number of values, then $f_1(x)$ is bounded, hence $f(x) - f_1(x) \geqq 2u_n$ for $x \in e_n$ and all large n. Therefore

$$\int_S \Phi(|f - f_1|)d\mu \geqq \int_{e_n} \Phi(2u_n)d\mu = \delta_n\Phi(2u_n) > 1.$$

But this implies [5, p. 80, Theorem 2] that $\|f - f_1\|_{L_\Phi} \geqq 1$, and proves our assertion.

We can now prove our main result:

THEOREM 2. *Let L_Φ be an Orlicz space for the measure space (S, B, μ) and let $l = \mu S$. Then L_Φ is equal to a space $\Lambda(W)$ if and only if Φ satisfies for some $\delta > 0$*

$$\int_{\Psi^{-1}(l^{-1})}^{\infty} \frac{\Psi(\delta x)}{\Psi(x)^2} d\Psi(x) < +\infty. \tag{11}$$

PROOF. First we assume that $\Phi(u)$ satisfies (10). In view of Theorem 1 we have to prove the following. If $W_0(e)$ is defined by (7), then $\Lambda(W_0) \supset L_\Phi$ is true if and only if (11) is satisfied.

This inclusion $\Lambda(W_0) \supset L_\Phi$ means that each $f \in L_\Phi$, $f(x) \geqq 0$, satisfies $\|f\|_{\Lambda(W_0)} < +\infty$ or equivalently

$$\begin{aligned} \int_0^\infty W(f^{-1}(y))dy &= \int_0^\infty \Phi_0(\mu f^{-1}(y))dy \\ &= \int_0^\infty dy \int_0^{\mu f^{-1}(y)} \Phi_0'(u)du = \int_0^l f^*(u)\Phi_0'(u)du < +\infty. \end{aligned} \tag{12}$$

Here $f^*(u)$ denotes the decreasing rearrangement of $f(x)$, that is a decreasing positive function defined on $0 < u < l$, equimeasurable with $f(x)$ with respect to the Lebesgue measure on $(0, l)$. Because the measure μ on S is full-valued, for each f^* on $(0, l)$ we can find an equimeasurable positive function f on S. Hence, if f runs through all positive functions of $L_\Phi(S)$, f^* will run through all decreasing $f^* \in L_\Phi(0, l)$. Since $\Phi_0'(u)$ is decreasing, (12) is equivalent to $\int_0^l f\Phi_0' du$

$<+\infty$ for all positive $f\in L_\Phi(0, l)$. But $f\in L_\Phi(0, l)$ is equivalent [**4**, p. 80] with $\int_0^l\Phi(\delta f)du<+\infty$ for some $\delta>0$. Thus condition (12) becomes $\int_0^l f\Phi_0' du<+\infty$ for all $f\geqq 0$ with $\int_0^l\Phi(f)du<+\infty$ or [**4**, p. 138] simply $\Phi_0\in L_\Psi^0(0, l)$. Using again (9), we find with $y=\Psi^{-1}(1/u)$, $t=\Psi'(y)$, $y=\phi(t)$,

$$\Phi_0'(x) = \frac{\Phi(t)}{t} \geqq \frac{1}{M}\,\frac{\Phi(2t)}{t} \geqq \frac{1}{Mt}\int_t^{2t}\phi(u)du$$
$$\geqq \frac{\phi(t)}{M} = \frac{1}{M}\,\Psi^{-1}\left(\frac{1}{u}\right).$$

Hence $\Phi_0' \in L_\Psi(0, l)$ is equivalent to

$$\int_0^l \Psi\left(\delta\Psi^{-1}\left(\frac{1}{u}\right)\right)du < +\infty \qquad \text{for some } \delta > 0. \tag{13}$$

Making the substitution $x=\Psi^{-1}(u^{-1})$ we see that this is equivalent to (11).

We complete the proof by showing that (11) implies (10). Assume that (10) does not hold. Let for example $\Phi(2u)/\Phi(u)$ be unbounded for $u\to\infty$. Then also $\phi(2u)/\phi(u)$ is unbounded, for an integration of $\phi(2u)\leqq M\phi(u)$ would give $\Phi(2u)/\Phi(u)\leqq M_1$ for large u. There exists therefore a sequence u_n with

$$\phi(2u_n) \geqq n\phi(u_n), \qquad\qquad u_n \to \infty.$$

Putting $v_n=\phi(u_n)$, $v_n' =\phi(2u_n)$, we find $v_n' \geqq nv_n$ and $\psi(v_n') = 2\psi(v_n)$. Let $\delta>0$ be chosen, then for $v_n'/2\leqq v\leqq v_n'$ and all large n,

$$\Psi(\delta v) \geqq \Psi\left(\frac{1}{2}\,\delta v_n'\right) \geqq \left(\frac{1}{2}\,\delta v_n' - v_n\right)\psi(v_n)$$
$$\geqq \frac{1}{3}\,\delta v_n'\psi(v_n) = \frac{1}{6}\,\delta v_n'\psi(v_n')$$
$$\geqq \frac{1}{6}\,\delta\Psi(v).$$

It is clear that $\Psi(v_n') \geqq 2\Psi(v_n'/2)$. Hence

$$\int_{v_n'/2}^{v_n'} \frac{\Psi(\delta u)}{\Psi(u)^2}\,d\Psi(u) \geqq \frac{1}{6}\,\delta\log\frac{\Psi(v_n')}{\Psi\left(\frac{1}{2}\,v_n'\right)} \geqq \frac{1}{6}\,\delta\log 2.$$

This shows that (11) is violated for all $\delta>0$. The case when (10) is not satisfied for $u \to 0$ is handled in the same way. The proof is complete.

There is also a theorem in the opposite direction.

THEOREM 3. *Let* $\Lambda(W)$ *with* $W(e) = \Phi_0(\mu e)$ *correspond to a measure space* (S, B, μ), *and let* $\Phi_0(u)$ *be an increasing continuous concave function for* $u \geqq 0$ *with* $\Phi_0(0) = 0$. *Let* $\Phi_0(u) = uP(u)$, $u > 0$. *Then* $\Lambda(W)$ *is a space* L_Φ *if and only if*

$$\int_0^l \frac{dx}{P^{-1}(\delta P(x))} < +\infty, \qquad \text{for some } \delta > 0. \tag{14}$$

This follows easily from Theorem 2; we leave the details to the reader.

Summing up Theorems 2 and 3 we see that the relation $L_\Phi = \Lambda(W)$ holds only in very exceptional cases. The integral (11) diverges for $\delta = 1$; hence condition (11) means that Ψ is very rapidly increasing. Thus $\Psi(u) = e^{\log^p u}$, $p > 1$ and $\Psi(u) = e^{\log u \operatorname{loglog} u} = u^{\operatorname{loglog} u}$ satisfy (11). For the space L^1, $\Phi(u) = 1$ for $u > 0$, and $\Psi(v) = \infty$ for $v > 1$. We can say that the relation $L_\Phi = \Lambda(W)$ can hold only if these spaces are fairly close to L^1.

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