ON THE FINITE DIMENSIONALITY OF EVERY IRREDUCIBLE UNITARY REPRESENTATION OF A COMPACT GROUP

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We shall prove that every irreducible unitary representation of a compact group is finite dimensional. Our argument is a variation of known proofs and it hardly could be based on an idea different from those already current. It makes no use of the Peter-Weyl theorem and of compact or Hilbert-Schmidt operators and seems simpler than the proofs in [**1**; **2**; **3**; **4**]. Its crucial point is that the prospectively finite dimension of the representation Hilbert space is expressible by a known integral formula.

Let $\mathcal{H}\neq 0$ be a Hilbert space and $x\rightarrow U_x$ be a group homomorphism of a compact group G into the group $U(\mathcal{H})$ of all unitary operators in $\mathcal{H}$, such that the scalar product $(\xi | U_x\eta)$ is a continuous function of $x\in G$ for all $\xi, \eta\in\mathcal{H}$. Suppose that this representation is irreducible, namely that there is no closed vector subspace of $\mathcal{H}$ invariant under all U_x except the trivial ones 0 and $\mathcal{H}$. Then there results that $\mathcal{H}$ is finite dimensional. In fact, let $\xi, \eta, \xi', \eta'\in\mathcal{H}$. Denoting complex conjugation by a star, since

$$\left|\int (\xi | U_x\eta)\cdot(\xi' | U_x\eta')^* dx\right| \leqq \|\xi\|\cdot\|\eta\|\cdot\|\xi'\|\cdot\|\eta'\|,$$

there is an operator T on $\mathcal{H}$ depending on η, η' such that

$$\int (\xi | U_x\eta)\cdot(\xi' | U_x\eta')^* dx = (T\xi | \xi').$$

T commutes with every U_t since

$$\begin{aligned}(TU_t\xi | \xi') &= \int (\xi | U_{t^{-1}x}\eta)\cdot(\xi' | U_x\eta')^* dx \\ &= \int (\xi | U_x\eta)\cdot(\xi' | U_{tx}\eta')^* dx = (T\xi | U_t^*\xi') = (U_tT\xi | \xi'),\end{aligned}$$

from which $TU_t = U_tT$ follows. The irreducibility of the representation then implies that T is a scalar operator, that is $T=\lambda(\eta, \eta')I$ and we get

$$\int (\xi | U_x\eta)\cdot(\xi' | U_x\eta')^* dx = \lambda(\eta, \eta')(\xi | \xi').$$

Received by the editors February 29, 1960.

By interchanging the roles of the couples (ξ, ξ') and (η, η') and using the rule $\int f(x^{-1})dx = \int f(x)dx$, we get

$$\lambda(\eta, \eta')(\xi \mid \xi') = \lambda(\xi, \xi')^*(\eta \mid \eta')^*.$$

Hence $\lambda(\eta, \eta') = c(\eta \mid \eta')^*$, where c is a constant, and

$$(1) \qquad \int (\xi, U_x\eta)\cdot(\xi' \mid U_x\eta')^* dx = c(\xi \mid \xi')\cdot(\eta \mid \eta')^*.$$

If we let ξ, η, ξ', η' all become equal to a unit vector α, we get

$$c = \int \mid (\alpha \mid U_x\alpha) \mid^2 dx.$$

Hence $c > 0$, since the positive continuous function whose integral is c has strictly positive value at the identity of G.

Now let $e_1, \cdots, e_n$ be orthonormal in $\mathcal{H}$. Let η, η' become equal to e_i and ξ, ξ' become equal to α in (1). By adding the resulting equalities and using Bessel's inequality

$$(2) \qquad \sum_{i=1}^{n} (\alpha \mid U_x e_i)\cdot(\alpha \mid U_x e_i)^* \leqq \|\alpha\|^2 = 1$$

since $U_x e_1, \cdots, U_x e_n$ are orthonormal, we get $nc \leqq 1$, that is $n \leqq 1/c$. This completes the proof that the dimension of $\mathcal{H}$ is finite.

We remark that, if n is supposed to be the finite dimension of $\mathcal{H}$, then (2) holds as an equality and so we get $nc = 1$, that is $c = 1/n$. Then (1) becomes a known formula (see [5, Chapter V]) which we took as motivation for the above proof.

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