A SET IS A 3 CELL IF ITS CARTESIAN PRODUCT WITH AN ARC IS A 4 CELL[1]

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We are learning strange things about cartesian products.

Shapiro has shown that the cartesian product of a line and a certain example described by Whitehead [16] is topologically E^4. The example is a bounded open subset of E^3 that is simply connected but which contains a simple closed curve which does not lie in any topological cube in it [3]. Glimm has given a proof of this result in [9] and a different proof is suggested in [5].

Artin and Fox have given [8] an example of a wild arc in S^3 whose complement is simply connected though topologically different from E^3. Kirkor has shown [11] that the cartesian product of this complement and a line is E^4.

Bing has described a nonmanifold [2] whose cartesian product with a line is E^4 [4; 5]. A modification of this example gives a compact nonmanifold whose cartesian product with a circle is topologically the same as the cartesian product of a 3 sphere and a circle. The modification is a monotone decomposition of S^3 whose set of nondegenerate elements is a Cantor set of tame arcs.

Rosen has complicated the example of Bing to show [14] that there is a set which is not locally a manifold at any point but whose cartesian product with a line is E^4. A proof that the cartesian product of a line and such a decomposition is E^4 has also been given by K. W. Kwun.

Fox and Artin described [8] an arc in E^3 whose complement is not simply connected. Curtis has shown [7] that the cartesian product of a line and the decomposition of E^3 whose only nondegenerate element is this arc is E^4. He later genralized this result to show that any decomposition of E^n whose only nondegenerate element is an arc gives E^{n+1} under a cartesian product with a line. One might wonder about any monotone decomposition of E^3 each of whose nondegenerate elements is an arc.

Poénaru gives [12] an example of a 4 manifold with boundary different from a cell whose cartesian product with an arc is a 5 cell. One might wonder just how far these unexpected things go.

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Question. Is there a complex different from a manifold whose cartesian product with a line is a manifold? The existence of such a complex would show that there is a triangulation of a manifold such that the star of a vertex is not a cell.

Raymond has shown [**13**] that $A\times B=M$ is a generalized manifold with possible boundary over a coefficient domain L if and only if A and B are generalized manifolds over L.

The following theorem shows another direction in which the pathologies are limited.

THEOREM. *A set is a* 3 *cell if its cartesian product with an interval is a* 4 *cell.*

PROOF. Suppose $X\times[0, 1]=I^4$. We give a sequence of lemmas exhibiting enough properties of X to insure that it is a 3 cell.

It follows from Lemma 1 that X can be embedded in E^3. Hence, we suppose that X lies in E^3 and use Bd X and Int X to denote the point set boundary and the interior respectively of the set X in E^3.

We show that Bd X is a 2 sphere. This follows from the Kline sphere characterization [**1**] since Bd X is connected (Lemma 9), locally connected (Lemma 10) and is not separated by any pair of its points (Lemma 12) but is separated by each simple closed curve in it (Lemma 13). Dyer has pointed out to me that instead of using the Kline sphere characterization, one can use results of Wilder [**17**; **18**] instead.

Since Int X is uniformly locally simply connected (Lemma 11), Bd X is tame from the inside [**6**]. This insures that X is a 3 cell.

Suppose M is a set whose cartesian product with an interval $[0, 1]$ is an n cell I^n. In the following lemmas we note certain properties of M.

LEMMA 1. *M can be embedded in E^{n-1}.*

PROOF. We show that each of $M\times 0$ and $M\times 1$ is contained in the $(n-1)$ sphere Bd I^n. Suppose $p\in M$ and $0<\epsilon<1$. If $p\times\epsilon$ were a point of the interior of the n cell $M\times[0, \epsilon]$, then $(p\times[\epsilon, 1])+(M\times[0, \epsilon])$ would be an n cell with a feeler sticking out of its interior. The invariance of domain theorem [**10**, p. 96] implies that such a set cannot be imbedded in I^n. Hence $p\times\epsilon$ is a point of Bd$(M\times[0, \epsilon])$. Since there is a homeomorphism of $M\times[0, \epsilon]$ onto $M\times[0, 1]$ taking $p\times\epsilon$ onto $p\times 1$, $p\times 1$ is a point of Bd I^n. Similarly $p\times 0$ is a point of Bd I^n.

Since M can be imbedded in E^{n-1}, we suppose that it lies there. In the following lemmas we used Bd M and Int M to denote the point set boundary and interior respectively of M in E^{n-1}.

LEMMA 2. *M is of dimension $n-1$ at each of its points.*

PROOF. It is not of dimension more than $n-1$ at any point since it lies in E^{n-1}. It is not of dimension less than $n-1$ at any point or else $I^n = M \times [0, 1]$ would be of dimension less than n somewhere.

LEMMA 3. *M is the closure of an open subset of E^{n-1}.*

PROOF. If it were not such a subset, it would be of dimension less than $n-1$ at some point.

LEMMA 4. Bd $M \times (0, 1) \subset$ Bd I^n.

PROOF. Suppose $p \in$ Bd M and N is a spherical neighborhood of p in E^{n-1}. Then $p \times \epsilon$ $(0 < \epsilon < 1)$ is a limit point of $(N \times (0, 1)) - (M \times (0, 1))$. It follows from the invariance of domain that $p \times \epsilon$ does not lie in any open n manifold in $M \times (0, 1)$. Hence $p \times \epsilon$ is not a point of Int I^n.

LEMMA 5. Int $M \times (0, 1) \subset$ Int I^n.

This result follows from the invariance of domain theorem. We summarize Lemmas 1, 4, 5 as follows:

$$\text{Bd}\, I^n = (M \times 0) + (M \times 1) + (\text{Bd}M \times (0, 1)),$$

$$\text{Int}\, I^n = \text{Int}\, M \times (0, 1).$$

We recall certain definitions. Let F_t $(0 \leqq t \leqq 1)$ be a one parameter (the parameter is t) family of maps of a set A into B such that if F_t is regarded as taking $A \times t$ into B, F_t $(0 \leqq t \leqq 1)$ takes $A \times [0, 1]$ continuously into B. Then F_t $(0 \leqq t \leqq 1)$ is called a *homotopy pulling F_0 to F_1*. We are particularly interested in the case where $F_1(A)$ is a point. In this case we say F_1 is a *constant map*.

A map f of A into B is *inessential* if there is a homotopy of A into B pulling f to a constant map. If there is no such homotopy, f is called an *essential map*.

A subset A of B *can be shrunk to a point in B* if the cone over A can be mapped into B in such a way that the map is the identity on the base of the cone. (This cone lies in an abstract space and is not supposed to intersect B except at its base.) Hence, A can be shrunk to a point in B if the identity map of A into B is inessential.

A set is *contractible* if the identity map of the set into itself is inessential.

A set A is *uniformly locally connected in dimension n* if for each $\epsilon > 0$, there is a $\delta > 0$ such that each map of an n sphere S^n into a δ subset of A can be extended to map an $(n+1)$-ball bounded by S^n

into an ϵ subset of A. We say that the map of S^n into a δ subset of A can be shrunk to a constant on an ϵ subset of A.

In determining whether or not M is uniformly locally connected it is irrelevant which metric we use for M since M is compact. However, there is more of a problem when we deal with Int M. We suppose that M is given a prescribed metric and that this metric is the same on Int M. We suppose that the metric of $M\times[0, 1]=I^n$ is given by the metric of a cube in E^n rather than by the cartesian metric.

LEMMA 6. *M is uniformly locally connected in all dimensions.*

PROOF. This is because $M\times[0, 1]$ has these properties. To see that each map f of an m sphere S^m into a small subset of M can be shrunk to a constant map on a small subset of M, consider the map f' of S^m into $M\times 1/2$ given by $f'(p)=f(p)\times 1/2$. Consider a homotopy F_t $(0\leqq t\leqq 1)$ shrinking f' to a constant in a small subset of I^n. Let g be the projection of I^n onto M that takes $p\times\epsilon$ to p. Then gF_t $(0\leqq t\leqq 1)$ is a homotopy shrinking the map f to a constant on a small subset of M.

LEMMA 7. *M is contractible.*

The proof of Lemma 7 is similar to the proof of Lemma 6.

LEMMA 8. Bd $M\times 1/2$ *irreducibly separates* $M\times 0$ *from* $M\times 1$ *on* Bd I^n.

PROOF. Each point $p\times 1/2$ of Bd $M\times 1/2$ is needed to separate $M\times 0$ from $M\times 1$ on Bd I^n since $p\times[0, 1]$ is an arc on Bd I^n from $M\times 0$ to $M\times 1$.

LEMMA 9. Bd *M is connected.*

PROOF. This follows from Lemma 8 and the unicoherence of the $(n-1)$ sphere Bd I^n.

LEMMA 10. Bd *M is uniformly locally connected in all dimensions.*

PROOF. The proof is the same as the proof of Lemma 6 except that we define F_t to be a homotopy of S^m into Bd $I^n-((M\times 0)+(M\times 1))$.

REMARK. We do not prove that Bd M is homotopically trivial even in low dimensions. In fact Poénaru's example [12] shows that Bd M need not even be simply connected in the case where $n=5$.

LEMMA 11. Int *M is uniformly locally connected in all dimensions.*

PROOF. We are to prove that for each $\epsilon>0$ there is a $\delta>0$ such that each map of an m sphere S^m into a δ subset of Int M can be shrunk to a constant map on an ϵ subset of Int M.

Let h be the homeomorphism of M onto $M\times 1/2$ given by $h(p)=p\times 1/2$ and g the projection of I^n onto $M\times 1/2$ given by $g(p\times t)=p\times 1/2$. We suppose I^n has its ordinary Euclidean metric. Let $\epsilon_1>0$ be so small that each ϵ_1 subset of I^n has an image under $h^{-1}g$ of diameter less than ϵ. Let $\delta>0$ be so small that each δ subset of M has a diameter of less than ϵ_1 under h.

Let f be a map of S^m into a δ subset of Int M. Then hf takes S^m into an ϵ_1 subset of Int I^n. There is a homotopy F_t $(0\leqq t\leqq 1)$ shrinking hf to a constant map on an ϵ_1 subset of Int I^n. Then $h^{-1}gF_thf$ shrinks f to a constant map on an ϵ subset of Int M.

LEMMA 12. *No $(n-4)$ sphere in* Bd M *separates it.*

PROOF. Suppose T is a topological $(n-4)$ sphere in Bd M. Then $T\times(0, 1)$ is an $(n-3)$ manifold in Bd $M\times(0, 1)$. However, Bd $M\times(0, 1)$ is a connected open subset of Bd I^n and is hence a connected $(n-1)$ manifold. Since no $(n-3)$ manifold even locally separates an $(n-1)$ manifold, T does not separate Bd M.

LEMMA 13. *Each topological $(n-3)$ sphere T in* Bd M *separates it.*

PROOF. Consider $(M\times 0)+(M\times 1)+(T\times(0, 1))=X$. We show in Lemma 14 that there is an essential map of X onto an $(n-2)$ sphere. This shows [10] that X separates the $(n-1)$ sphere Bd I^n and that Bd $I^n-X=(\text{Bd } M-T)\times(0, 1)$ is not connected. Hence, Bd $M-T$ is not connected.

Lemma 13 can also be proved by using the methods of homology theory by showing that $(M\times 0)+(M\times 1)+(T\times(0, 1))$ contains an $(n-2)$ Čech cycle that does not bound in $(M\times 0)+(M\times 1)+(T\times(0, 1))$.

The proof of the following lemma was supplied by a conversation with E. E. Floyd and Walter Rudin.

LEMMA 14. *Suppose A_1, A_2 are two closed sets whose intersection is an n sphere S^n such that S^n can be shrunk to a point in each A_i. Then there is an essential map of A_1+A_2 onto an $(n+1)$ sphere.*

PROOF. Let C_1, C_2 be two $(n+1)$ cells sewed together along their boundaries to form an $(n+1)$ sphere. Let f be a map of A_1+A_2 into C_1+C_2 that takes the n sphere $A_1\cdot A_2$ homeomorphically onto Bd $C_1=\text{Bd } C_2=C_1\cdot C_2$, A_1 into C_1, and A_2 into C_2.

We show that f is essential by showing that there is no homotopy F_t $(0\leqq t\leqq 1)$ of A_1+A_2 into C_1+C_2 that shrinks f to a constant map.

We show that the assumption that there is such a homotopy F_t leads to the contradiction that the identity map of C_1+C_2 onto itself is inessential.

Since $A_1 \cdot A_2$ can be shrunk to a point in A_1, there is a map g of C_1 into A_1 that is f^{-1} in taking $C_1 \cdot C_2$ onto $A_1 \cdot A_2$. Similarly, the map g can be extended to take C_2 into A_2. We call the extended map g.

Consider the map fg. It is homotopic to the identity since it takes each C_i $(i=1, 2)$ onto itself and is the identity on Bd $C_1=$ Bd C_2. The assumption that f is inessential leads to the contradiction that $F_t g$ $(0 \leqq t \leqq 1)$ is a homotopy pulling fg to a constant map.

It may be noted that the proof of Lemma 14 did not use the fact that $A_1 \cdot A_2$ is an n sphere but merely that $A_1 \cdot A_2$ contains an n sphere that is a retract of $A_1 \cdot A_2$.

A homological analogue of Lemma 14 gives that if A_1, A_2 are compact and $A_1 \cdot A_2$ contains an n cycle which bounds in each of A_1, A_2 but not in $A_1 \cdot A_2$, then A_1+A_2 contains an $(n+1)$ cycle that does not bound in A_1+A_2.

THEOREM. *Each factor of an n cell I^n, $n \leqq 4$, is a cell.*

PROOF. Szumbarski proved [15] this result for $n \leqq 3$.

Szumbarski also proved [15] that each 1 dimensional factor of an n cell is a 1 cell.

Young showed [19] that each 2 dimensional factor of an n cell is a 2 cell.

Young's proof of Theorem 1 in [19] showed that if one factor of an n cell is of dimension $n-1$, the other factor contains no triod. Hence, if a factor of I^4 is 3 dimensional, the other factor is an arc and it follows from our first theorem that the 3 dimensional factor is a 3 cell.

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