NOTE ON A TOPOLOGY OF A DUAL SPACE

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1. **Introduction.**[2] Let G be a locally compact group, and let L be a Lie group. Let us denote by D the set of all continuous homomorphisms from G into L. By introducing the so-called compact-open topology in D, D becomes a complete uniform space. We shall call the uniform space D the *dual space* of G with respect to L in this paper.

When G is abelian and L is the group of rotations in the euclidean plane, D coincides with the character group in the sense of Pontrjagin, and it is well known that D is also locally compact. Here we would like to generalize the proposition to the nonabelian case. The purpose of this note is to prove the following two theorems:

THEOREM 1. *Let G be a locally compact group, and L a compact Lie group. Then the dual space D is locally compact.*

THEOREM 2. *Let G be a locally compact group and L a Lie group. If there is a compact generating system in G, then the dual space D is locally compact.*

REMARK. If G is an infinitely generated free group with the discrete topology, and if L is a noncompact Lie group, then D is homeomorphic with an infinite product of copies of L and accordingly D is not locally compact.

2. **Preliminaries.**[2] Let M be a set and H a topological group. Let us denote by $(M \to H)$ the topological space composed of all functions from M into H relative to the product topology (=finite-open topology). By defining the multiplication: $(fg)(x) = f(x)g(x)$ for f, $g \in (M \to H)$ and $x \in M$, $(M \to H)$ becomes a topological group. When M is a topological space, we consider a subset (subgroup) $[M \to H]$ of $(M \to H)$ composed of all continuous functions. For a compact subset C of M and a neighborhood E of the identity in H we define a subset

$$(C, E) = \{f \mid f \in [M \to H] \text{ and } f(C) \subset E\} \text{ of } [M \to H].$$

It is easy to see that the set of all possible (C, E)'s forms a base for a neighborhood system of the identity of a topological group. After

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[2] For the first two sections please refer to the books listed in the end of the paper.

this $[M\to H]$ denotes the topological group thus topologized. Clearly the identity mapping brings $[M\to H]$ continuously into $(M\to H)$.

Let us assume that M is a topological group. Let D' be the set of all homomorphisms from M into H. We define D by $D=D'\cap[M\to H]$. Since an element f of D' is characterized by the property $f(x)f(y)=f(xy)$ for any pair x, y of M, D' is a closed subspace of $(M\to H)$. D is closed in $[M\to H]$ similarly. When H is a complete topological group $(M\to H)$ and $[M\to H]$ are known to be complete, and so D' and D are complete. Hence D' and D are complete uniform spaces. It is to be noted that when H is locally compact $f(x)$ is a jointly continuous function from $[M\to H]\times M$ into H.

PROPOSITION 1. *Let G and L be locally compact groups. Then* (1) *$(G\to L)$ and $[G\to L]$ are complete topological groups, and D' and D are closed subspaces of $(G\to L)$ and $[G\to L]$ respectively.* (2) *For an f in $[G\to L]$ $f(x)$ is jointly continuous from $[G\to L]\times G$ into L.* (3) *When L is compact, D' is compact.*

NOTATIONS. We shall write 1 for the identity of the group in question. By a *nucleus* of a locally compact group we mean an open symmetric neighborhood of 1, whose closure is compact. Let G and L be locally compact groups, and D the dual space of G with respect to L. Let f be in D. For a compact subset C of G and a nucleus E of L we put

$$(f;C,E)=\{g\mid g\in D \text{ and } (f^{-1}g)(C)\subset E\}.$$

The set of all possible $(f;C,E)$'s of course forms a base for the neighborhood system of f in D.

Let L be a Lie group of dimension r. Let us introduce a canonical system of coordinates of the first kind in a suitable nucleus of L. By changing the scale if necessary, we may identify a nucleus E with an open euclidean sphere of radius 2: for a in E, there corresponds coordinates $(a_1,\cdots,a_r)$ so that $\|a\|=(a_1^2+\cdots+a_r^2)^{1/2}<2$. We denote $S(\delta)=\{a\mid \|a\|<\delta\}$ for $0<\delta\leqq 2$. By the definition of a canonical coordinate system of the first kind, $ta=(ta_1,\cdots,ta_r)$ forms a local one-parameter subgroup for $-2/\|a\|<t<2/\|a\|$, and if j is a positive integer less than $2/\|a\|$, then $a^j=ja$.

PROPOSITION 2. *Let m be a positive integer. Then an element a of L is contained in the sphere $S(1/m)$ if and only if $a, a^2,\cdots,a^m\in S(1)$.*

PROOF. If $a\in S(1/m)$, namely if $\|a\|<1/m$, then for $j\leqq m$, $j<1/\|a\|<2/\|a\|$, and so $a^j=ja$ and $\|a^j\|=j\|a\|<1$.

Conversely if a is in $S(1)$ but not in $S(1/m)$, namely if $1/m\leqq\|a\|$

<1, and if j is the first integer so that $1 \leqq j\|a\|$, then $j \leqq m$ and $j\|a\| < 2$, whence $a^j = ja \notin S(1)$.

3. **Proof of Theorem** 1. Let G be a locally compact group, and L a compact Lie group. We retain the notations in 2. We know that $(G \to L)$ is compact, and the identity mapping I is continuous and one-to-one from D into $(G \to L)$. Let f be an element of D. Since f is continuous we can find a nucleus V of G so that $f(V) \subset S(1/2)$. Take a nucleus E of L so that $S(1/2)\bar{E} \subset S(1)$, where $\bar{E}$ denotes the closure of E. Let us prove that the neighborhood $(f;\ \bar{V},\ E)$ has a compact closure in D. Let us put $F = \{g \mid g \in D \text{ and } (f^{-1}g)(\bar{V}) \subset \bar{E}\}$. It clearly suffices to prove that

(A) I is an open mapping in F, and

(B) $I(F)$ is closed in $(G \to L)$.

PROOF OF (A). Let g be in F, and let $(g;\ C_1,\ E_1)$ be a given neighborhood of g. Let us find a finite subset $\{x_1, \cdots, x_k\}$ of G and a positive integer m so that $(g;\ \{x_1, \cdots, x_k\},\ S(1/m)) \cap F \subset (g;\ C_1,\ E_1)$. For this purpose take a sufficiently large m such that $S(1/m)^3 \subset E_1$, and take a nucleus U of G such that $U^m \subset V$. Since C_1 is compact we can find a finite subset $\{x_1, \cdots, x_k\}$ of G so that $x_1U \cup \cdots \cup x_kU \supset C_1$. We shall prove that $\{x_1, \cdots, x_k\}$ and m thus obtained satisfy the requirement.

Let us take an h in $(g;\ \{x_1, \cdots, x_k\},\ S(1/m)) \cap F$. Let x be in C_1. Then x can be written in a form $x = x_jy$ where $y \in U$ and $1 \leqq j \leqq k$. Hence $g(x)^{-1}h(x) = (g(x_j)g(y))^{-1}h(x_j)h(y) = g(y)^{-1}(g(x_j)^{-1}h(x_j))h(y)$. Since h is in $(g;\ \{x_1, \cdots, x_k\},\ S(1/m))$, we have

$$g(x_j)^{-1}h(x_j) \in S(1/m).$$

Next for $n = 1, 2, \cdots, m$, $y^n \in V$ and so $f^{-1}(y^n)g(y^n) \in \bar{E}$, and $f(y^n) \in S(1/2)$. Hence $g(y^n) \in S(1/2)\bar{E} \subset S(1)$. Because $g(y^n) = g(y)^n$ we have $g(y) \in S(1/m)$ by Proposition 2. Similarly $h(y) \in S(1/m)$. Hence $g(x)^{-1}h(x) \in S(1/m)^3 \subset E_1$, namely $h \in (g;\ C_1,\ E_1)$.

PROOF OF (B). Let $\mathrm{Cl}(I(F))$ be the closure of $I(F)$ in $(G \to L)$. Since D' is closed in $(G \to L)$ we have $\mathrm{Cl}(I(F)) \subset D'$. Let g be an element of $\mathrm{Cl}(I(F))$ and let E_2 be a given nucleus of L. We may take n so large that $S(1/n) \subset E_2$. Take a nucleus W of G so that $W^{n+1} \subset V$. Then for an x in W, $x, x^2, \cdots, x^{n+1} \in V$ and so $f(x^i)^{-1}h(x^i) \in \bar{E}$ for $h \in F$ and $i = 1, 2, \cdots, n+1$. Therefore $h(x^i) = h(x)^i \in S(1/2)\bar{E} \subset S(1)$. Hence by Proposition 2, $h(x) \in S(1/(n+1))$. Since $g \in \mathrm{Cl}(I(F))$, $g(x) \in \mathrm{Cl}(S(1/(n+1))) \subset S(1/n) \subset E_2$, namely $g(W) \subset E_2$, which implies that $g \in I(D)$. Thus we proved that $\mathrm{Cl}(I(F)) \subset I(D)$, and so $I(F)$ is closed.

4. **Proof of Theorem** 2. Let G be a locally compact group with a compact generating system. Then obviously there is a nucleus V which generates G. Let us consider a function J from D into $[\overline{V}\to L]$ defined by $Jf=f|\overline{V}$, where $f|\overline{V}$ is the restriction of f in $\overline{V}$.

LEMMA 1. *J is bicontinuous, namely D is homeomorphic with $J(D)$.*

PROOF. Since V generates G, J is one-to-one. The continuity is obvious. Let us prove the openness of J.

Let $(f;\ C_1,\ E_1)$ be a given neighborhood of f in D. We can find a positive integer m such that $V^m\supset C_1$. Let us consider a function ϕ defined by

$$\phi(u_1,\cdots,u_m;\ a_2,\cdots,a_m)$$
$$= a_m^{-1}(\cdots(a_3^{-1}((a_2^{-1}u_1a_2)u_2)a_3u_3)\cdots)a_mu_m,$$

where $u_i\in L$ and $a_i\in f(\overline{V})$. Since $\phi(1,\cdots,1;\ a_2,\cdots,a_m)=1$, we can find a nucleus E_2 of L so that if $u_1,\cdots,u_m$ are in E_2, then $\phi(u_1,\cdots,u_m,\ a_2,\cdots,a_m)$ is in E_1. Let x be an arbitrary element in C_1. Then we can find $x_1,\cdots,x_m$ in V so that $x=x_1\cdots x_m$. Let g be in $(f;\ \overline{V},\ E_2)$. Then

$$\begin{aligned} f(x)^{-1}g(x) &= (f(x_1)\cdots f(x_m))^{-1}(g(x_1)\cdots g(x_m)) \\ &= \phi(u_1,\cdots,u_m;\ a_2,\cdots,a_m) \end{aligned}$$

where $f(x_j)^{-1}g(x_j)=u_j\in E_2$ and $f(x_j)=a_j\in f(\overline{V})$. Hence $f(x)^{-1}g(x)\in E_1$, namely $(f;\ \overline{V},\ E_2)\subset(f;\ C_1,\ E_1)$, which proves the lemma.

Next let us fix an element f of D, and for a g in D we define g^* in $[G\to L]$ by $g^*(x)=f(x)^{-1}g(x)$. Then g^* satisfies

$$g^*(xy) = f(y)^{-1}g^*(x)f(y)g^*(y) \tag{1}$$

for x and y in G, and conversely if a function g^* in $[G\to L]$ satisfies (1), then $g^*=f^{-1}g$ for some g in D. Let W be an arbitrary nucleus of G. Set

$$F = f^{-1}(f;\overline{W},\ \mathrm{Cl}(S(1/2))) = \{g^*\mid g^*(\overline{W})\subset \mathrm{Cl}(S(1/2)) \text{ and } g^*\in f^{-1}D\}.$$

LEMMA 2. *$F|\overline{W}=\{g^*|\overline{W}|g^*\in F\}$ is an equicontinuous family of functions.*

PROOF. Let E be a given nuclus of L. Let us prove the existence of a nucleus U of G so that $x,\ xy\in\overline{W},\ y\in U$ and $g^*\in F$ imply that $g^*(x)^{-1}g^*(xy)\in E$. For this purpose let us first take an integer m, with $S(1/m)\subset E$, and a positive number ϵ such that

$$\mathrm{Cl}(S(1/2))S(\epsilon)^2\subset S(1). \tag{2}$$

Next for the ϵ and the m we shall take a positive number δ_1 such that if $a, b \in S(1)$ and $c \in S(\delta_1)$ then

$$(a^{-1}c^{-1}acb)^j \in b^j S(\epsilon) \qquad \text{for } j = 1, 2, \cdots, m. \tag{3}$$

Next let us find a positive number δ_2 such that if $a \in S(1)$ and $c \in S(\delta_2)$ then

$$(c^{-j+1}ac^{j-1})(c^{-j+2}ac^{j-2}) \cdots (c^{-1}ac)a \in a^j S(\epsilon) \tag{4}$$

for $j=1, 2, \cdots, m$, and let δ be the minimum of δ_1 and δ_2. Take a nucleus U_1 of G, with $f(U_1) \subset S(\delta)$, and also take a nucleus U_2 such that $U_2^m \subset W$, and set $U_1 \cap U_2 = U$. Then we have

$$f(U) \subset S(\delta) \quad \text{and} \quad U^m \subset W. \tag{5}$$

Let x and xy be in $\overline{W}$ so that $y \in U$. Let g^* be in F. Then $g^*(\overline{W}) \subset \mathrm{Cl}(S(1/2)) \subset S(1)$. By (1) we have

$$g^*(x)^{-1}g^*(xy) = g^*(x)^{-1}f(y)^{-1}g^*(x)f(y)g^*(y). \tag{6}$$

Hence using (3) we have

$$(g^*(x)^{-1}g^*(xy))^j \in g^*(y)^j S(\epsilon) \qquad \text{for } j = 1, 2, \cdots, m. \tag{7}$$

On the other hand, since

$$g^*(y^j) = (f(y)^{-j+1}g^*(y)f(y)^{j-1}) \cdots (f(y)^{-1}g^*(y)f(y))g^*(y),$$

(4) implies that

$$g^*(y^j) \in g^*(y)^j S(\epsilon). \tag{8}$$

From (7) and (8) we have

$$(g^*(x)^{-1}g^*(xy))^j \in g^*(y^j)S(\epsilon)^2.$$

From $y^j \in W$ it follows that $g^*(y^j) \subset \mathrm{Cl}(S(1/2))$, whence

$$(g^*(x)^{-1}g^*(xy))^j \in \mathrm{Cl}(S(1/2))S(\epsilon)^2 \subset S(1)$$

by (2), for $j=1, 2, \cdots, m$. Hence by Proposition 2

$$g^*(x)^{-1}g^*(xy) \in S(1/m) \subset E.$$

LEMMA 3. *$F|\overline{W}$ is compact, if $\overline{W}$ generates G.*

PROOF. We shall denote by $(\overline{W} \to \mathrm{Cl}(S(1/2)))$ the topological space composed of all functions from $\overline{W}$ into $\mathrm{Cl}(S(1/2))$ with the product topology, and let us consider the identity mapping I from $F|\overline{W}$ into $(\overline{W} \to \mathrm{Cl}(S(1/2)))$, which is clearly one-to-one and continuous. Because $(\overline{W} \to \mathrm{Cl}(S(1/2)))$ is compact, it suffices to prove that I is an open mapping and $I(F|\overline{W})$ is a closed set.

(A) Openness of I. Let g^* be an element of F. For a neighborhood of g^* in $F|\overline{W}$ we can find a smaller one of the form $(g^*; \overline{W}, E) = f^{-1}(g; \overline{W}, E) = \{h^* | h^* \in f^{-1}D \text{ and } (g^{*-1}h^*)(\overline{W}) \subset E\}$, where E is a nucleus of L. Let us find a finite subset $\{x_1, \cdots, x_k\}$ of $\overline{W}$ and a nucleus E_1 of L such that $(g^*; \overline{W}, E) \supset (g^*; \{x_1, \cdots, x_k\}, E_1) \cap F$, where $(g^*; \{x_1, \cdots, x_k\}, E_1) = f^{-1}(g; \{x_1, \cdots, x_k\}, E_1)$.

For this purpose let us first take a nucleus E_1 of L satisfying $E_1 a^{-1} E_1 a E_1 \subset E$ for $a \in f(\overline{W})$, and find a nucleus U of G in $\overline{W}$ so that $F(U) \subset E_1$ using Lemma 2. Next let us take $x_1, \cdots, x_k \in \overline{W}$ so that $x_1 U \cup \cdots \cup x_k U \supset \overline{W}$.

Let x be in $\overline{W}$. Then we can find a y in U such that $x = x_j y$ for some j. Let h^* be in $(g^*; \{x_1, \cdots, x_k\}, E_1) \cap F$. Then by (1) $g^*(x)^{-1}h^*(x) = g^*(x_j y)^{-1}h^*(x_j y) = g^*(y)^{-1}f(y)^{-1}(g^*(x_j)^{-1}h^*(x_j))f(y)h^*(y)$. On the other hand, since $y \in U$ we have $g^*(y) \in E_1$ and $h^*(y) \in E_1$, and $h^* \in (g^*; \{x_1, \cdots, x_k\}, E_1)$ implies that $g^*(x_j)^{-1}h^*(x_j) \in E_1$. From $U \subset \overline{W}$ it follows that $f(U) \subset f(\overline{W})$. Hence $g^*(x)^{-1}h^*(x) \subset E$.

(B) $I(F|\overline{W})$ is closed.

For a pair x, y of elements of $\overline{W}$ so that xy is also in $\overline{W}$, and for a g^* in F we have the relation (1). Hence if h^* is in the closure of the image of F in $(\overline{W} \to \overline{S}(1/2))$, then $h^*(xy) = f(y)^{-1}h^*(x)f(y)h^*(y)$. Let us put $h(x) = f(x)h^*(x)$ for $x \in \overline{W}$. Then $h(xy) = h(x)h(y)$ for $x, y, xy \in \overline{W}$. Since $F|\overline{W}$ is equicontinuous by Lemma 2, h^* is continuous, and so is h.

Let $x_1, \cdots, x_m, y_1, \cdots, y_n$ be elements of $\overline{W}$. If $x_1 \cdots x_m = y_1 \cdots y_n$, then for $g = fg^* \in fF$,

$$g(x_1) \cdots g(x_m) = g(y_1) \cdots g(y_n),$$

whence we have

$$h(x_1) \cdots h(x_m) = h(y_1) \cdots h(y_n).$$

Because $\overline{W}$ generates G, h can be extended to a continuous homomorphism $\tilde{h}$ from G into L.

Hence $h^* = f^{-1}\tilde{h} \in F$, and $\tilde{h}^* = h^*$ in $\overline{W}$. Therefore $I(F|\overline{W})$ is closed.

Proof of Theorem 2. By Lemma 1, D is homeomorphic with $J(D)$. On the other hand if we put $W = V$ in Lemma 3 then we have the result that $J(F)$ is compact. Since $[\overline{V} \to L]$ is a topological group, $J(f)J(F) = J(f; \overline{V}, \mathrm{Cl}(S(1/2)))$ is also compact. Accordingly $J(D)$ is locally compact, and so is D.

References

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