FAMILIES OF SOLUTIONS OF A PERTURBATION PROBLEM

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1. **Introduction.** In previous papers [2; 3], topological degree theory was used to establish the existence of periodic solutions of systems of ordinary differential equations with a nonlinear perturbation term. In [2; 3], methods for computing the appropriate topological degree were developed and in [3], it was shown that for 2-dimensional, totally degenerate systems, the absolute value of the topological degree, call it $|d|$, is a lower bound for the number of distinct periodic solutions in the following sense: if the perturbation term in the system of differential equations contains a term which is a function of the independent variable only (such a term is often called a forcing term) then if this term is varied arbitrarily slightly, the number of distinct periodic solutions is greater than or equal to $|d|$.

In this note, we extend and sharpen this result. First we show that the result holds for systems of arbitrary dimension and with arbitrary degree of degeneracy. Secondly, we show that these periodic solutions form families continuous in the perturbation parameter.

As in [3], we start from the treatment of perturbed systems given by Coddington and Levinson [1, pp. 356 ff.]. This note is independent of [2; 3] and may be regarded as an extension of the theory developed in [1, pp. 356 ff.].

2. **Families of solutions.** We consider the n-dimensional system,

$$\dot{x} = Ax + \mu f(x, t, \mu), \tag{E}$$

where A is a constant matrix, μ is a parameter, and f has period 2π in variable t. We study the degenerate case, i.e., the case in which the linear part of (E) [the equation $\dot{x}=Ax$] has q nonzero solutions of period 2π where $1 \leqq q \leqq n$. We assume that f has continuous second derivatives in all values of x, t, μ. Let $x(t, \mu, c)$ be the general solution of (E) such that $x(0, \mu, c)=c$. Because of the uniqueness condition in the general existence theorem, the condition that $x(t, \mu, c)$ have period 2π is:

$$x(2\pi, \mu, c) - c = 0,$$

or using the variation of constants formula,

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$$(1)\qquad (e^{2\pi A} - E)c + \mu \int_0^{2\pi} e^{(2\pi - s)A} f[x(s, \mu, c), s, \mu] ds = 0,$$

where E is the identity matrix. (Equation (1) is Equation (3.20) on p. 360 in Coddington and Levinson [1].) Thus the problem of finding periodic solutions of (E) is that of solving the system (1) of n equations for the n unknowns, $c_1, \cdots, c_n$, the components of c.

Following [1] we make the following assumption.

ASSUMPTION 1. Matrix A has the canonical form:

$$A = \begin{bmatrix} A_1 & & & & & & & \\ & \ddots & & & & & & \\ & & A_k & & & & & \\ & & & B_1 & & & & \\ & & & & \ddots & & & \\ & & & & & B_m & \\ & & & & & & C \end{bmatrix}$$

where the elements not shown are zeros. Each A_j, $j = 1, \cdots, k$, is a matrix of α_j (α_j even) rows and columns of the form

$$A_j = \begin{bmatrix} S_j & & & \\ E_2 & S_j & & \\ & \ddots & \ddots & \\ & & E_2 & S_j \end{bmatrix}$$

where all the elements are zero except S_j and E_2, and

$$S_j = \begin{pmatrix} 0 & -N_j \\ N_j & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where N_j is a positive integer. (A matrix A_j may have only two rows and two columns in which case $A_j = S_j$.) Each matrix B_j has β_j rows and columns, $j = 1, \cdots, m$, and is of the form

$$B_j = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & & 1 & 0 \end{bmatrix}$$

where B_j may have only one row and column in which case B_j consists of the single element 0. The matrix C has $(n - \sum_{j=1}^{k} \alpha_j - \sum_{j=1}^{m} \beta_j)$

rows and columns and has no characteristic roots of the form iN for any integer N including $N=0$. Matrix C need not be in canonical form.

If $(c_1, \cdots, c_i, \cdots, c_n)$ is an n-vector, the indices i corresponding to the last two rows of any A_j or to the last row of any B_j are called exceptional indices. They are indices with the following form:

$$i = \alpha_1 + \alpha_2 + \cdots + (\alpha_j - 1),$$
$$i = \alpha_1 + \alpha_2 + \cdots + \alpha_j,$$

where $j=1, \cdots, k$, and

$$i = \alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_j$$

where $j=1, \cdots, m$. The indices i corresponding to the first two rows of any A_j or to the first row of any B_j are called singular indices. They are indices with the following form:

$$\begin{aligned} i = 1, 2, \alpha_1 + 1, \alpha_2 + 2, \cdots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1} + 1, \\ \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1} + 2, \alpha_1 + \cdots + \alpha_k + 1, \\ \alpha_1 + \cdots + \alpha_k + \beta_1 + 1, \cdots, \alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots \\ + \beta_{m-1} + 1. \end{aligned}$$

There are $(2k+m)$ exceptional indices and $(2k+m)$ singular indices. The number $q=2k+m$ is the degree of degeneracy of the problem. Throughout this paper, we assume that $q>0$, i.e., that there is at least one A_j or one B_j in the canonical form of matrix A.

As described in [1], if the $c_1, \cdots, c_n$ are relabelled $c_1', \cdots, c_n'$ so that the first $(n-q)c_i'$'s are the components of $(c_1, \cdots, c_n)$ with nonexceptional indices and the last qc_i''s are the components with exceptional indices and if j, j' denote singular and nonsingular indices, then equation (1) may be replaced with:

$$(2) \qquad \begin{aligned} N(c_1', \cdots, c_{n-q}') + \mu\left(\int_0^{2\pi} e^{(2\pi-s)A} f[x(s, \mu, c), s, \mu]ds\right)_{(j')} &= 0, \\ \left(\int_0^{2\pi} e^{(2\pi-s)A} f[x(s, \mu, c), s, \mu]ds\right)_{(j)} &= 0, \end{aligned}$$

where N is a nonsingular $(n-q)\times(n-q)$ matrix acting on vector $(c_1', \cdots, c_{n-q}')$ and $(\int_0^{2\pi} e^{(2\pi-s)A} f[x(s, \mu, c)s, \mu]ds)_{j'}$ denotes the vector composed of the $(n-q)$ components of $\int_0^{2\pi} e^{(2\pi-s)A} f[x(s, \mu, c), s, \mu]ds$ which have nonsingular indices. Similarly

$$\left(\int_0^{2\pi} e^{(2\pi-s)A} f[x(s, \mu, c), s, \mu]ds\right)_{(j)}$$

denotes the vector composed of the q components which have singular indices.

The left side of (2) defines a continuous mapping (call it $\mathfrak{M}_\mu$) of real Euclidean n-space into itself. Let c'' denote the vector c in which the $(n-q)$ components with nonexceptional indices have been set equal to zero. (The nonzero components of c'' are $c'_{n-q+1}, \cdots, c'_n$.) Then

$$\left(\int_0^{2\pi} e^{(2\pi-s)A} f[e^{sA}c'', s, 0]ds\right)_{(j)}$$

defines a continuous mapping of real q-space into itself which we call M.

LEMMA 1. *If* $(c^0_{n-q+1}, \cdots, c^0_n)$ *is a solution of*

$$M(c'_{n-q+1}, \cdots, c'_n) = 0 \tag{3}$$

at which the Jacobian of M *(denote it by* $J(M)$*) is* $\neq 0 [=0]$, *then* $(0, \cdots, 0, c^0_{n-q+1}, \cdots, c^0_n)$ *is a solution of*

$$\mathfrak{M}_0(c'_1, \cdots, c'_n) = 0 \tag{4}$$

at which $J(M_0) \neq 0$ $[=0]$. *Conversely any solution of* (4) *has the form* $(0, \cdots, 0, c^0_{n-q+1}, \cdots, c^0_n)$ *where* $(c^0_{n-q+1}, \cdots, c^0_n)$ *is a solution of* (3), *and if* $J(M_0) \neq 0$ $[=0]$ *at* $(0, \cdots, 0, c^0_{n-q+1}, \cdots, c^0_n)$, *then* $J(M)$ $\neq 0$ $[=0]$ *at* $(c^0_{n-q+1}, \cdots, c^0_n)$.

PROOF. Follows from inspection of systems (3) and (4).

LEMMA 2. *Let* $\mathcal{S}$ *be a sphere in n-space with center at the origin and radius r, and let S be a sphere in q-space with center at the origin and radius r. If the topological degree of M at O (the origin in q-space) and relative to S exists, then if* μ *is sufficiently small, the topological degree of* $\mathfrak{M}\mu$ *at O (the origin of the n-space) and relative to S exists and equals the topological degree of M.*

PROOF. Follows from the definition of topological degree and the fact that the topological degree is invariant under homotopy.

Now we make the following assumption:

ASSUMPTION 2. Let $f_1(x, t, \mu), \cdots, f_n(x, t, \mu)$ denote the components of $f(x, t, \mu)$. We assume that if j is a singular index, then

$$f_j(x, t, \mu) = g_j(x, t, \mu) + h_j(t)$$

where $h_j(t)$ is a function of t only with a continuous second derivative and $h_j(t)$ has period 2π in t.

Now let $h(t)$ be the vector whose jth component is 0 if j is nonsingular and whose jth component is $h_j(t)$ if j is singular. Similarly define the vector $g(t)$. If Assumption 2 is satisfied, then system (4) has the form:

$$N(c_1', \cdots, c_{n-q}') = 0,$$

$$\left(\int_0^{2\pi} e^{(2\pi-s)A} g[e^{sA}c, s, 0]ds\right)_{(j)} = -\left(\int_0^{2\pi} e^{(2\pi-s)A} h(s)ds\right)_{(j)}, \tag{5}$$

where j denotes a singular index. We want to write the constant terms on the right in (5) in detail. First, these terms may be written:

$$-\left(\int_0^{2\pi} e^{sA} h(-s)ds\right)_{(j)}.$$

From the definition of singular index, the terms may then be written:

$$\begin{aligned} -\int_0^{2\pi} [\cos(N_1 s)h_1(-s) - \sin(N_1 s)h_2(-s)]ds &= b_1 \\ -\int_0^{2\pi} [\sin(N_1 s)h_1(-s) + \cos(N_1 s)h_2(-s)]ds &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots& \\ -\int_0^{2\pi} [h_q(-s)]ds &= b_q. \end{aligned} \tag{6}$$

The form of the last equation in system (6) implies that there is a B_j in the canonical form of matrix A. If there is no B_j in the canonical form, then all the equations in system (6) look like the first two equations in the system.

Now the point $b = (b_1, \cdots, b_q)$ is a point in q-space.

LEMMA 3. *Given $\epsilon_0 > 0$, an arbitrary positive number, then there exists a neighborhood N of b in q-space such that if $p \in N$, then there exist $h_1^{(1)}(t), \cdots, h_q^{(1)}(t)$, all of period 2π in t and such that*

$$\max_{0 \leq t \leq 2\pi} \left| h_j(t) - h_j^{(1)}(t) \right| < \epsilon_0, \qquad (j = 1, \cdots, q),$$

and such that if $h_j(t)$ is replaced by $h_j^{(1)}(t)$ in the integrals in (6), *$j = 1, \cdots, q$, the resulting values $b_1^{(1)}, \cdots, b_q^{(1)}$ are the coordinates of point p.*

PROOF. This follows from inspection of the integrals in (6). For example, we may take:

$$h_1^{(1)}(t) = h_1(t) + \frac{a}{\pi}\cos(N_1 t),$$

$$h_2^{(1)}(t) = h_2(t) + \frac{b}{\pi} \sin (N_1 t),$$

where a, b are arbitrary small constants.

DEFINITION. Given $\epsilon_0 > 0$, then functions $h_1(t), \cdots, h_q(t)$ are said to be varied less than ϵ_0 if they are replaced by functions $h_1^{(1)}(t), \cdots, h_n^{(1)}(t)$, all of period 2π in t, and all having continuous second derivatives, and such that

$$\max_{0 \leqq t \leqq 2\pi} \left| h_j(t) - h_j^{(1)}(t) \right| < \epsilon_0.$$

Now we need a lemma about topological degree which was shown in a previous paper [4] to be an easy consequence of a theorem due to Sard [5].

LEMMA 4. *Let J be a continuous map defined on the closure $\overline{R}$ of a region $R \subset R^n$ and differentiable in R. Suppose that the topological degree of J at point p_0 is $d \neq 0$. Then there is a neighborhood U of p_0 and a set E of n-dimensional measure zero, $E \subset U$, such that $p \in U - E$ implies that $J^{-1}(p)$ is a finite set consisting of at least $|d|$ points.*

PROOF. See [4, Lemma 3.2].

In the two lemmas and the theorem which follow, Lemma 4 will be applied. Functions $h_1(t), \cdots, h_q(t)$ will be varied less than ϵ_0 and in such a way as to avoid a set of zero measure of the kind referred to in Lemma 4. For brevity, we shall refer to this kind of variation of $h_1(t), \cdots, h_q(t)$ simply as varying $h_1(t), \cdots, h_q(t)$ less than ϵ_0.

LEMMA 5. *If the topological degree at O of M relative to sphere S is $d \neq 0$, then if $h_1(t), \cdots, h_q(t)$ are varied less than ϵ_0, an arbitrary positive number, then the system*

$$M(c'_{n-q+1}, \cdots, c'_n) = 0 \tag{3}$$

has exactly m solutions inside sphere S, where $m \geqq |d|$ and $J(M) \neq 0$ at each solution in S of (3).

PROOF. Follows directly from Lemmas 3 and 4.

LEMMA 6. *If the topological degree of M at O and relative to S is $d \neq 0$, and if ϵ_0 is an arbitrary positive number, then there exists $\mu_1 > 0$ such that if $h_1(t), \cdots, h_q(t)$ are varied less than ϵ_0, then for $|\mu| < \mu_1$, the system*

$$\mathfrak{M}_\mu(c'_1, \cdots, c'_n) = 0$$

has m distinct solutions

$$(c_1^{(i)}(\mu), \cdots, c_n^{(i)}(\mu)), \qquad (i = 1, \cdots, m).$$

Each solution is a continuous function of μ *for* $|\mu| < \mu_1$. *Also* $(c_1^{(i)}(\mu), \cdots, c_n^{(i)}(\mu))$ *is in the interior of* $\mathcal{S}$ *for* $i=1, \cdots, m$ *and for* $|\mu| < \mu_1$. *The solutions are distinct in the following sense: if* $|\mu_2| < \mu_1$, $|\mu_3| < \mu_1$ *and if* $i_1 \neq i_2$, *then*

$$c_j^{(i_1)}(\mu_2) \neq c_j^{(i_2)}(\mu_3), \qquad (j = 1, \cdots, n).$$

Proof. From Lemmas 1 and 5, it follows that

$$\mathfrak{M}_0(c_1', \cdots, c_n') = 0 \tag{7}$$

has m solutions at which $J(M_0) \neq 0$. Also (7) has no other solutions by Lemma 1. The conclusion of the lemma then follows from Lemma 2 because the topological degree is the sum of the signs of the Jacobians. (Each solution $(c_1^{(i)}(\mu), \cdots, c_n^{(i)}(\mu))$ is a continuous function of μ by the implicit function theorem.)

Theorem. *If the topological degree of* M *at* O *and relative to* S *is* $d \neq 0$ *and if* ϵ_0 *is an arbitrary positive number, then there exists* $\mu_1 > 0$ *such that if* $h_1(t), \cdots, h_q(t)$ *are varied less than* ϵ_0, *equation* (E) *has* $m(\geqq |d|)$ *distinct periodic solutions* $x_i(t, \mu, c)$ $(i=1, \cdots, m)$ *where* c *is in the interior of* $\mathcal{S}$, *solution* $x_i(t, \mu, c)$ *depends continuously on* μ *for* $|\mu| < \mu_1$ *and*

$$\lim_{\mu \to 0} x_i(t, \mu, c) = e^{tA}c^{(i)}$$

where $(c^i) = (c_1^{(i)}(0), \cdots, c_n^{(i)}(0))$ *for* $i=1, \cdots, m$.

Proof. Follows from the derivation of (1), and Lemma 6, and the general existence theorem for solutions of (E).

Remark. Note that the theorem gives no information about the existence or nonexistence of periodic solutions $x(t, \mu, c)$ such that c is outside S. If the radius of S is changed, then, in general, the value of μ_1 (see Lemma 6) is changed.

Remark. If (E) is totally degenerate, i.e., if $n=q$, the proof of the theorem goes through in the same way except that Lemma 1 can be omitted and Lemma 2 is simpler.

Remark. If the topological degree of M at O and relative to S is zero, then the same kind of arguments as have been used here may be applied to show that if equation (3) has solutions in S, and if $h_1(t), \cdots, h_q(t)$ are varied less than ϵ_0, then (E) has a finite set of families $x_i(t, \mu, c)$ of periodic solutions as described in our theorem. But equation (3) may have no solutions in S. If the topological degree is zero, it is generally necessary to employ finer methods to obtain existence theorems (see [**3**]).

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QUOTIENT GROUPS OF REDUCED ABELIAN GROUPS

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Let G be a reduced torsion p-group. (In this paper, group will mean Abelian group.) Let B be a basic subgroup of G. It is well known that $|B|^{\aleph_0} \geqq |G|$, where $|S|$ denotes the cardinal of the set S. Fuchs gives a proof of this in [1, p. 102], and attributes it to Kulikov. This has turned out to be a very useful fact, and the purpose of this short note is to generalize it. Now, as is generally known, every torsion group G has a basic subgroup B; that is, a pure subgroup B that is a direct sum of cyclic groups, and such that G/B is divisible. To obtain such a B, simply take B_p to be a basic subgroup of the p-component of G, and let $B = \sum_p \oplus B_p$. It is easy to see that in this more general situation $|B|^{\aleph_0} \geqq |G|$ still holds, and in fact follows from the corresponding statement for p-groups. The generalization we will prove is the following

Theorem. *Let G be a reduced torsion group, and let H be a subgroup of G such that G/H is divisible. Then* $|H|^{\aleph_0} \geqq |G|$.

Proof. Our proof uses some homological results of Harrison in [2]. Notice that we may assume that $|H| < |G|$, that G is infinite, and hence that $|G/H| = |G|$. Let Q and Z denote the additive group of rationals and integers, respectively. From the exact sequence

$$0 \to H \to G \to G/H \to 0,$$

we get the exact sequence

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