ON FLEXIBLE POWER-ASSOCIATIVE ALGEBRAS OF DEGREE TWO[1]

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Let $\mathfrak{A}$ be a simple, flexible, power-associative algebra of degree two over an algebraically closed field $\mathfrak{F}$ of characteristic not equal to 2, 3 or 5. Then $\mathfrak{A}$ has a unity element [5, Theorem 3.5] $1=u+v$ where u and v are absolutely primitive orthogonal idempotents of $\mathfrak{A}$. The algebra $\mathfrak{A}$ can be decomposed as a vector space direct sum $\mathfrak{A}=\mathfrak{A}_u(2)+\mathfrak{A}_u(1)+\mathfrak{A}_u(0)$ where x is in $\mathfrak{A}_u(\lambda)$ if and only if $ux+xu=\lambda x$.[2] If $\mathfrak{A}_u(\lambda)\mathfrak{A}_u(1)+\mathfrak{A}_u(1)\mathfrak{A}_u(\lambda)\subseteq\mathfrak{A}_u(1)$ for $\lambda=2, 0$, u is called a stable idempotent and $\mathfrak{A}$ is said to be u-stable. If $\mathfrak{A}$ is u-stable for every idempotent u in $\mathfrak{A}$ then $\mathfrak{A}$ is called a stable algebra.

In §1 it is shown that if $\mathfrak{A}$ is a u-stable algebra then there is an element w in $\mathfrak{A}_u(1)$ such that $w^2=1$. The existence of this element is used in §2 to develop some of the multiplicative properties of $\mathfrak{A}$. Finally in §3 it is proved that every simple, flexible, stable, power-associative algebra of degree two over an algebraically closed field $\mathfrak{F}$ of characteristic $\neq 2, 3, 5$ is a noncommutative Jordan algebra.

1. Let $\mathfrak{A}$ be a simple, flexible, power-associative algebra of degree two over an algebraically closed field $\mathfrak{F}$ of characteristic $\neq 2, 3, 5$. We have the linearization of the power-associative identity

$$\sum (xy + yx)(zw + wz) = \sum [(xy + yx)z]w \text{(symmetric in } x, y, z \text{ and } w) \tag{1}$$

and the flexible identity

$$(xy)z + (zy)x = x(yz) + z(yx). \tag{2}$$

The algebra $\mathfrak{A}$ has an attached algebra $\mathfrak{A}^+$ which is the same vector space as $\mathfrak{A}$ but has a product $x \circ y$ defined by $x \circ y=(1/2)(xy+yx)$ where xy is the product of $\mathfrak{A}$. The idempotent u of $\mathfrak{A}$ is an idempotent of $\mathfrak{A}^+$ and $\mathfrak{A}_u(\lambda)=\mathfrak{A}_u^+(\lambda)$. Since u and v are absolutely primitive $\mathfrak{A}_u^+(2)=u\mathfrak{F}+\mathfrak{N}_2$ and $\mathfrak{A}_u^+(0)=v\mathfrak{F}+\mathfrak{N}_0$ where $\mathfrak{N}_2$ and $\mathfrak{N}_0$ are subalgebras of nilpotent elements of $\mathfrak{A}^+$ [4, Proof of Theorem 3]. The notations $\mathfrak{N}$ shall be used in the sum $\mathfrak{N}_2+\mathfrak{N}_0$ of the two algebras $\mathfrak{N}_2$ and $\mathfrak{N}_0$;

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[2] Since Albert's result [2, Theorem 5, page 562] on the idempotent decomposition of a flexible ring is basis to all of our results no further reference shall be made to it each time it is used.

$\mathfrak{C}$ for the sum $\mathfrak{A}_u(2)+\mathfrak{A}_u(0)$ and $z=u-v$. We first prove that $\mathfrak{N}_2$, and consequently $\mathfrak{N}_0$, is a subalgebra of $\mathfrak{A}$.

LEMMA 1.1. *If* $x\in\mathfrak{A}_u(2)$ *and* $w\in\mathfrak{A}_u(1)$ *then* $(xw)_1=2[u(w\circ x)]_1=2(x\circ uw)_1$ *and* $(wx)_1=2(x\circ wu)_1=2[(w\circ x)u]_1$.[3] *(We use the symbol* x_λ *to denote the* $\mathfrak{A}_u(\lambda)$ *component of the element* x.*)*

PROOF. By the flexible identity (2) we have $(xw)u+(uw)x=x(wu)+u(wx)$ and $u(xw)+wx=xw+(wx)u$. Adding these identities we get $2u\circ xw+wx+(uw)x=xw+x(wu)+2u\circ wx$. But $2(u\circ xw)_1=(xw)_1$ and $2(u\circ wx)_1=(wx)_1$. Therefore $[(uw)x]_1=[x(wu)]_1$. Now $wu=w-uw$. Therefore $[(uw)x]_1=(xw)_1-[x(uw)]_1$ and $2(x\circ uw)_1=(xw)_1$. Similarly $2(x\circ wu)_1=(wx)_1$. We also have $[(xw)u]_1=[u(wx)]_1$. Adding $[u(xw)]_1$ to both sides gives us $(xw)_1=2[u(x\circ w)]_1$.

If $\mathfrak{N}_2$ is not a subalgebra of $\mathfrak{A}_u(2)$ we can find two elements x and y in $\mathfrak{N}_2$ such that $xy=u+n$ and $yx=-u+n'$ with n and n' in $\mathfrak{N}_2$. Let a be an arbitrary element of $\mathfrak{A}_u(1)$. From (2) we have $ua+na+(ay)x=x(ya)+an'-au$ and $a=x(ya)+an'-(ay)x-na$. Together with Lemma 1.1 this gives us $a=4[x\circ u(y\circ ua)]_1+2[n'\circ au]_1-4[x\circ(y\circ au)u]_1-2[n\circ ua]_1=4[x\circ(y\circ u(ua))]_1+2[n'\circ au]_1-4[x\circ(y\circ(au)u)]_1-2(n\circ ua)_1$. Therefore $\mathfrak{A}_u(1)\subseteq[\mathfrak{N}_2\circ\mathfrak{A}_u(1)]_1$. Now each element x of $\mathfrak{A}_u(2)$ corresponds to a linear transformation S_x on $\mathfrak{A}_u(1)$ to $\mathfrak{A}_u(1)$ such that $S_x\colon w\rightarrow(x\circ w)_1$. S_x is nilpotent when x is nilpotent [3, Lemma 1]. The associative enveloping algebra of the linear transformations S_x is nilpotent [1, Theorem 8]. We can now write $a=2[u(ua)S_yS_x+2(au)S_{n'}-2[(au)u]S_yS_x-2(ua)S_n$. Since a is an arbitrary element of $\mathfrak{A}_u(1)$, a similar expression can be obtained for the elements $u(ua)$, au, $(au)u$ and ua of $\mathfrak{A}_u(1)$. Therefore by an inductive argument we have that a is a sum of terms of the form $wS_{x_1}S_{x_2}\cdots S_{x_t}$ where $w\in\mathfrak{A}_u(1)$, $x_i\in\mathfrak{N}_2$ and t is arbitrary. But t can be chosen large enough so that all products $S_{x_1}\cdots S_{x_t}=0$. Therefore $a=0$ and $\mathfrak{A}_u(1)=0$. But then $\mathfrak{A}=\mathfrak{A}_u(2)\oplus\mathfrak{A}_u(0)$ contradicting the assumption that $\mathfrak{A}$ is simple. Therefore we have

THEOREM 1.2. $\mathfrak{A}_u(2)=u\mathfrak{F}+\mathfrak{N}_2$ *where* $\mathfrak{N}_2$ *is a subalgebra in* $\mathfrak{A}$ *of nilpotent elements of* $\mathfrak{A}_u(2)$.

Our algebra shall be restricted further by imposing the condition that u be a stable idempotent; i.e., $\mathfrak{A}$ is u-stable. With this added assumption we prove the existence of an element w in $\mathfrak{A}_u(1)$ such that $w^2=1$. We need the following lemmas.

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[3] To avoid excessive use of parentheses we shall use the convention that the multiplicative operation in $\mathfrak{A}$ takes precedence over the multiplicative operation of $\mathfrak{A}^+$.

LEMMA 1.3. *If s and t are in $\mathfrak{A}_u(1)$ then $(st)_2=2(us\circ t)_2$ and $(ts)_0=2(us\circ t)_0$.*

PROOF. By the flexible identity (2) we have $t(su)+u(st)=(ts)u+(us)t$ and $ts-t(us)+u(st)=(ts)u+(us)t$. Therefore $ts+u(st)=(ts)u+2(us)\circ t$. Equating $\mathfrak{A}_u(2)$ components we get $(st)_2=2(us\circ t)_2$. Equating $\mathfrak{A}_u(0)$ components we get $(ts)_0=2(us\circ t)_0$.

LEMMA 1.4. *The set $\mathfrak{J}_2$ of all elements of $\mathfrak{A}_u(2)$ of the form $\sum(ts)_2$ is an ideal of $\mathfrak{A}_u(2)$ and the set $\mathfrak{J}_0$ of all elements of $\mathfrak{A}_u(0)$ of the form $\sum(ts)_0$ is an ideal of $\mathfrak{A}_u(0)$ for t and s in $\mathfrak{A}_u(1)$.*

PROOF. Let $y=v$, $x\in\mathfrak{A}_u(2)$, $w=t$ and $z=s$ be elements of $\mathfrak{A}_u(1)$ in (1) to get $(xt)s+(tx)s+s(xt)+s(tx)+(xs)t+(sx)t+t(xs)+t(sx) = [(xt+tx)s]v+[(xt+tx)v]s+[(xs+sx)t]v+[(xs+sx)v]t+(tx)s+(ts)x+(st)x+(sx)t$. The $\mathfrak{A}_u(2)$ component of each of the above terms, except possibly $(ts+st)x$, is clearly in $\mathfrak{J}_2$. Therefore, by equating components, the $\mathfrak{A}_u(2)$ component of $(ts+st)x$ is also in $\mathfrak{J}_2$. Since t and s are arbitrary elements of $\mathfrak{A}_u(1)$, $(ut\circ s)_2x$ is in $\mathfrak{J}_2$ and by Lemma 1.3, $2(ut\circ s)_2x=(ts)_2x=[(ts)x]_2$ is in $\mathfrak{J}_2$. Now $x(ts)+s(tx)=(xt)s+(st)x$. Therefore $[x(ts)]_2$ is also in $\mathfrak{J}_2$. We show the second part of the lemma in a similar manner.

Now assume that for every s, t of $\mathfrak{A}_u(1)$ we have $st\in\mathfrak{N}_2+\mathfrak{N}_0+\mathfrak{A}_u(1)$. Then $ts\in\mathfrak{N}_2+\mathfrak{N}_0+\mathfrak{A}_u(1)$ and $t\circ s\in\mathfrak{N}_2+\mathfrak{N}_0$. Since $\mathfrak{A}$ is u-stable we have that $\mathfrak{J}_2+\mathfrak{A}_u(1)+\mathfrak{J}_0$ is an ideal of $\mathfrak{A}$ for $\mathfrak{J}_2$ and $\mathfrak{J}_0$ defined as in Lemma 1.4. But this ideal of $\mathfrak{A}$ is contained in $\mathfrak{N}_2+\mathfrak{A}_u(1)+\mathfrak{N}_0$. Therefore the ideal must be zero since $\mathfrak{A}$ is simple. But this implies that $\mathfrak{A}_u(1)=0$ which contradicts the simplicity of $\mathfrak{A}$. We can conclude that there is a pair of elements s and t in $\mathfrak{A}_u(1)$ such that $st\notin\mathfrak{N}_2+\mathfrak{A}_u(1)+\mathfrak{N}_0$. This implies that either $(st)_2$ is not in $\mathfrak{N}_2$ or $(st)_0$ is not in $\mathfrak{N}_0$. By Lemma 1.3 we then have that either $(t\circ us)_2\notin\mathfrak{N}_2$ or $(ut\circ s)_0\notin\mathfrak{N}_0$ and therefore either $t\circ us$ or $ut\circ s$ is not in $\mathfrak{N}_2+\mathfrak{A}_u(1)+\mathfrak{N}_0$. Hence there is a pair of elements s and t in $\mathfrak{A}_u(1)$ such that $s\circ t\notin\mathfrak{N}_2+\mathfrak{N}_0$. Since $2s\circ t=(s+t)^2-s^2-t^2$, there is an element y in $\mathfrak{A}_u(1)$ such that y^2 is nonsingular. We can now state Albert's result [4, Lemma 3] as

THEOREM 1.5. *If y is a nonsingular element of $\mathfrak{A}_u(1)$ then there exists an x in $\mathfrak{F}(y)$, the algebra generated by y over $\mathfrak{F}$, such that $w=x\circ y$ is in $\mathfrak{A}_u(1)$ and $w^2=1$.*

From the existence of such an element w in $\mathfrak{A}_u^+(1)$ Albert has shown [4] that $\mathfrak{A}^+$ has the following properties.

LEMMA 1.6. *Let $\mathfrak{B}$ be the set of all elements b of $\mathfrak{C}$ such that $(b\circ w)\circ w=b$. Then $\mathfrak{B}^+$ is a subalgebra of the algebra $\mathfrak{C}^+$; both $\mathfrak{A}_u^+(2)$ and $\mathfrak{A}_u^+(0)$*

are isomorphic to $\mathfrak{B}^+$. $\mathfrak{A}_u(2) = u\mathfrak{B}$, $\mathfrak{A}_u(0) = v\mathfrak{B}$, $\mathfrak{C} = \mathfrak{B} + \mathfrak{B}z$, *and* $w \circ (c \circ w)$ *is in* $\mathfrak{B}$ *for every* c *in* $\mathfrak{C}$.

LEMMA 1.7. *Let* a *and* b *be in* $\mathfrak{B}$. *Then* $(w \circ a) \circ b = (w \circ b) \circ a = w \circ (a \circ b)$, $(w \circ a) \circ (w \circ b) = a \circ b$, $w \circ \mathfrak{B}z = 0$ *and* $w \circ (a \circ u) = w \circ (a \circ v) = (w \circ a)/2$.

LEMMA 1.8. $\mathfrak{A}_u(1) = w \circ \mathfrak{B} + \mathfrak{G}$ *where* $\mathfrak{G}$ *consists of all elements* g *of* $\mathfrak{A}_u(1)$ *such that* $w \circ g = 0$ *and if* $g \in \mathfrak{G}$, $b \in \mathfrak{B}$ *then there exists a* $d \in \mathfrak{B}$ *such that* $g \circ (z \circ b) = w \circ d$.

LEMMA 1.9. *The element* $e = (1 + w)/2$ *is an idempotent of* $\mathfrak{A}$ *and*

$$\mathfrak{A}_e(2) = e\mathfrak{B}, \qquad \mathfrak{A}_e(1) = z\mathfrak{B} + \mathfrak{G}.$$

2. In this section we shall develop multiplicative properties for A similar to those that Albert has developed for $\mathfrak{A}^+$.

LEMMA 2.1. *For every* c *in* $\mathfrak{C}$ *we have* $w(w \circ c) = (w \circ c)w \in \mathfrak{B}$.

PROOF. From (2) we have $w(wc) + cw^2 = w^2c + (cw)w$ and $w(cw) = (wc)w$. Adding these two identities and using the fact that $w^2 = 1$ we have $w(c \circ w) = (c \circ w)w = w \circ (c \circ w)$. Hence the lemma follows from Lemma 1.6.

LEMMA 2.2. *If* $x \in \mathfrak{A}_u(2)$ *and* $w \in \mathfrak{A}_u(1)$ *then* $u(wx) = (xw)u$. *If* $x \in \mathfrak{A}_u(0)$ *and* $w \in \mathfrak{A}_u(1)$ *then* $u(xw) = (wx)u$.

PROOF. By the flexible identity (2) with $x \in \mathfrak{A}_u(2)$ we have $u(xw) + wx = xw + (wx)u$. But $u(xw) = xw - (xw)u$ and $(wx)u = wx - u(wx)$. Therefore $(xw)u = u(wx)$. The second result of the lemma is obtained in a similar manner.

LEMMA 2.3. *If* $c \in \mathfrak{A}_u(2)$ *then* $2[(c \circ w)w]_2 = c$. *If* $b \in \mathfrak{B}$ *then*

$$2[(b_0 \circ w)w]_2 = b_2 = 2[(b_2 \circ w)w]_2.$$

PROOF. Let c be an element of $\mathfrak{C}$. Let $x = c$, $y = u$ and $z = w$ in (1). Using $w^2 = 1$ and Lemma 2.1 we have $4c_2 + 8(c \circ w)w = 4(c_2w)w + 2(wc)w + 2c + 4[(c \circ w)u]w + 4[(c \circ w)w]u$. From Lemma 2.2 we have $2(c \circ w)u = wc_2 + c_0w$. Therefore the above identity can be simplified to $4c_2 + 8(c \circ w)w = 8(c_2 \circ w)w + 4(w \circ c_0)w + 2c + 4[(c \circ w)w]u$. If $c = c_2$ then $c_0 = 0$ and $2c_2 = 4[(c_2 \circ w)w]_2$. If $c \in \mathfrak{B}$ then $(c \circ w)w = c$ and $4(c_0 \circ w)w = 2c$. Therefore $4(c_2 \circ w)w = 2c$. Equating $\mathfrak{A}_u(2)$ components we have $2[(c_0 \circ w)w]_2 = 2[(c_2 \circ w)w]_2 = c_2$.

LEMMA 2.4. *If* a *and* b *are elements of* $\mathfrak{B}$ *then* $wb = bw = b \circ w$ *and* $b(aw) = (ba) \circ w$ *and* $(wa)b = w \circ ab$.

PROOF. Let $x=y=w$ and $z=b$ in (1) to get $4(w\circ b)=wb+bw+[(bw+wb)w]w$ and $(w\circ b)=[(b\circ w)w]w$. But by Lemma 2.1 and the definition of B we have $[(b\circ w)w]w=bw$. Therefore $bw=wb=b\circ w$. This implies that $2b\circ(a\circ w)=b(aw)+(aw)b$. Since B^+ is a subalgebra of A^+ we also have that $2(a\circ b)\circ w=2(a\circ b)w=2w(a\circ b)$. But $2b\circ(a\circ w)=2(a\circ b)\circ w$ from Lemma 1.7. Combining these results with $(wa)b+(ba)w=w(ab)+b(aw)$ obtained from (2) we have $b(aw)=w\circ ba$ and $(wa)b=w\circ ab$.

THEOREM 2.5. *$\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$.*

PROOF. Using the elements wa, w and b in (2) we have $(wa)(wb)+b[w(wa)]=[(wa)w]b+(bw)(wa)$ and $(wa)(wb)+ba=ab+(bw)(wa)$. From Lemma 1.7 and Lemma 2.4 we have $(wa)(wb)=-(wb)(wa)+ab+ba$. Combining these results we get $(wa)(wb)=ab$ and $(wb)(wa)=ba$. Since $e=(1+w)/2$ is an idempotent of $\mathfrak{A}^+$ it is also an idempotent of $\mathfrak{A}$ and $\mathfrak{A}_e(2)$ is a subalgebra of $\mathfrak{A}$. From Lemma 1.9 the subspace $\mathfrak{A}_e(2)$ is equal to $e\mathfrak{B}$. Therefore every element of $\mathfrak{A}_e(2)$ can be written in the form $b+bw$ where $b\in\mathfrak{B}$. If a and $b\in\mathfrak{B}$ then there exists a $d\in\mathfrak{B}$ such that $(a+wa)(b+wb)=d+wd$. Multiplying out the left side and equating components in the decomposition of $\mathfrak{A}$ with respect to the idempotent u we have $2ab=d$. Therefore $ab\in\mathfrak{B}$. Hence $\mathfrak{B}$ is closed under the multiplication of $\mathfrak{A}$. The other properties of an algebra obviously hold in $\mathfrak{B}$.

It is easily seen from the above lemmas, in particular Lemma 2.3, that the mapping $b\rightarrow bu$ is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{A}_u(2)$. Similarly $\mathfrak{B}$ and $\mathfrak{A}_u(0)$ are isomorphic. Therefore we have

THEOREM 2.6. *$\mathfrak{A}_u(2)$ and $\mathfrak{A}_u(0)$ are isomorphic subalgebras of $\mathfrak{A}$.*

3. In addition to our previous assumptions on $\mathfrak{A}$ we shall now add the assumption that A is stable. We proceed with a sequence of lemmas leading to our main theorem that A is a noncommutative Jordan algebra.

LEMMA 3.1. *If $g\in\mathfrak{G}$ and $b\in\mathfrak{B}$ then gb and $bg\in\mathfrak{G}$.*

PROOF. Since $g\in\mathfrak{A}_u(1)$ and $b\in\mathfrak{A}_u(2)+\mathfrak{A}_u(0)$ we have $gb\in\mathfrak{A}_u(1)$. We can write $gb=wd+h$ where $d\in\mathfrak{B}$ and $h\in\mathfrak{G}$ by Lemma 1.8. But we also have $g\in\mathfrak{A}_e(1)$ and $b\in\mathfrak{A}_e(2)+\mathfrak{A}_e(0)$. Therefore $gb\in\mathfrak{A}_e(1)$ and we can write $gb=za+h'$ where $a\in\mathfrak{B}$ and $h'\in\mathfrak{G}$. Equating components of the two representations of gb we have $h=h'$, $wd=0$ and $za=0$. Hence $gb\in\mathfrak{G}$. In a similar manner we show that $bg\in\mathfrak{G}$.

LEMMA 3.2. *For all $d\in\mathfrak{B}$ we have $(wd)z=d\circ wz=w(dz)$.*

PROOF. From (2) we have $(zw)d+(dw)z=z(wd)+d(wz)$. Since d commutes with w and z anti-commutes with all elements of $\mathfrak{A}_u(1)$ we have $2(dw)z=d(wz)+(wz)d=2d \circ wz$. Also $(wd)z+(zd)w=w(dz)+z(dw)$. Since $w \circ dz=0$ we have $w(dz)=-(dz)w$ and $(zd)w=z(dw)$.

LEMMA 3.3. *For every a and $b\in\mathfrak{B}$ we have* $2[(aw)(bz)]z=(ab)w-(ba)w+[(wz)z] \circ d$ *where* $d=2a \circ b$.

PROOF. Let $x=bz$ and $y=a$ in (1) to obtain $4wd=dw+[(dz)w]z+2(ba)w+2(bw)a+2[(aw)z](bz)+2[(aw)(bz)]z+2[(az)(bz)]w+2[(az)w](bz)$. But from Lemmas 1.7 and 3.2 we have $-(az)w=w(az)=(wa)z=(aw)z$. Therefore the above identity can be reduced to $4wd=3(ab)w+5(ba)w+[(dz)w]z+2[(aw)(bz)]z$. From Lemma 3.2 we have $[(dz)w]z=-[w(dz)]z=-(d \circ wz)z$. By (2) we have $[d(wz)]z=-[z(wz)]d+d[(wz)z]+z[(wz)d]$. Therefore $2(d \circ wz)z=[d(wz)]z+[(wz)d]z=[(wz)d]z-[z(wz)]d+d[(wz)z]+z[(wz)d]=[(wz)z]d+d[(wz)z]=2d \circ [(wz)z]$. Hence $[(dz)w]z=-d \circ [(wz)z]$ and $2[(aw)(bz)]z=w(ab)-w(ba)+[(wz)z] \circ d$.

LEMMA 3.4. *For every pair of elements d and b in $\mathfrak{B}$ we have* $2(wd)(bz)=(bd+db) \circ wz+2wd \circ bz\in\mathfrak{S}$.

PROOF. From (2) we have $(wd)(bz)+z[b(wd)]=[(wd)b]z+(zb)(wd)$. Adding $(wd)(bz)$ to both sides and using Lemma 3.2 we have $2(wd)(bz)=[w(bd)]z+[w(db)]z+2(zb) \circ (wd)=(bd+db) \circ wz+2(zb) \circ (wd)$. We can apply [4, Lemma 10] to get $(zb) \circ (wd)$ in $\mathfrak{S}$. To show that $(bd+db) \circ wz$ is in $\mathfrak{S}$ we substitute w, w and z in (2) to get $w(wz)+zw^2=w^2z+(zw)w$. Since z anti-commutes with w we have $w \circ wz=0$. Therefore $wz\in\mathfrak{S}$ and by Lemma 3.1 $(bd+db) \circ wz\in\mathfrak{S}$.

LEMMA 3.5. *The set* $\mathfrak{M}=\mathfrak{N}+\mathfrak{A}_u(1)\mathfrak{N}+\mathfrak{N}\mathfrak{A}_u(1)$ *is an ideal of* $\mathfrak{A}$.

PROOF. Since $\mathfrak{N}$ is an ideal of $\mathfrak{C}$ we clearly have that $\mathfrak{A}\mathfrak{N}+\mathfrak{N}\mathfrak{A}\subseteq\mathfrak{M}$. Therefore it is sufficient to show that $\mathfrak{A}\mathfrak{J}+\mathfrak{J}\mathfrak{A}\subseteq\mathfrak{M}$ where $\mathfrak{J}=\mathfrak{A}_u(1)\mathfrak{N}+\mathfrak{N}\mathfrak{A}_u(1)\subseteq\mathfrak{A}_u(1)$. If h is any element of $\mathfrak{A}_u(1)$ we can find an α in $\mathfrak{F}$ such that $h+\alpha w$ is nonsingular. Therefore every h in $\mathfrak{A}_u(1)$ can be expressed as the difference of two nonsingular elements of $\mathfrak{A}_u(1)$. Hence it will be sufficient to show that all multiples of ha and ah are in $\mathfrak{M}$ for h in $\mathfrak{A}_u(1)$ and nonsingular and for a in $\mathfrak{N}$. Let h be any nonsingular element of $\mathfrak{A}_u(1)$ then by Theorem 1.5 there is a d in $\mathfrak{F}(h)\cap\mathfrak{C}$ such that $w=h \circ d$ is in $\mathfrak{A}_u(1)$ and $w^2=1$. Since the associative and commutative laws hold in $\mathfrak{F}(h)$ and d is nonsingular there is a $c\in\mathfrak{F}(h)\cap\mathfrak{C}$ such that $w \circ c=h$ and $(c \circ w) \circ w=c$. Therefore c is in the algebra $\mathfrak{B}$ corresponding to this w. Similarly, $d\in\mathfrak{B}$. Now let $b\in\mathfrak{B}$ then $(hb)z=[(wc)b]z=[w \circ cb]z=wz \circ cb$ by Lemmas 2.4 and 3.2.

Therefore if b is nilpotent $(hb)z$ is in $\mathfrak{J}$. Also $2[h(bz)]z=2[(wc)(bz)]z=(cb)w-(bc)w+[(wz)z]\circ(cb+bc)$ by Lemma 3.3. Again if b is nilpotent $[h(bz)]z$ is in $\mathfrak{J}$. But every element a of $\mathfrak{N}$ can be expressed in the form $b+b'z$ where b and b' are nilpotent elements of $\mathfrak{B}$. Therefore we have that $(ha)z$ is in $\mathfrak{J}$ for all nilpotent elements a in $\mathfrak{C}$ and all h in $\mathfrak{A}_u(1)$. A similar argument is used to show that $(ah)z$ is in $\mathfrak{J}$. Hence $z\mathfrak{J}=-\mathfrak{J}z\subseteq\mathfrak{J}$. Let x be any element of $\mathfrak{C}$. We can express x as $x=\alpha z+\beta 1+n$ where α and β are in $\mathfrak{F}$ and $n\in\mathfrak{N}$. Since $\mathfrak{J}\subseteq\mathfrak{A}_u(1)$ it is clear that $\mathfrak{N}\mathfrak{J}+\mathfrak{J}\mathfrak{N}\subseteq\mathfrak{J}$. Therefore we may conclude that for any $x\in\mathfrak{C}$ we have $x\mathfrak{J}+\mathfrak{J}x\subseteq\mathfrak{J}$.

It remains to show that $\mathfrak{A}_u(1)\mathfrak{J}+\mathfrak{J}\mathfrak{A}_u(1)\subseteq\mathfrak{M}$. Let $g\in\mathfrak{G}$ and hb as above. We have $(hb)z=[(wc)b]g=[w(cb)]g=[(cb)w]g=-(gw)(cb)+(cb)(wg)+g[w(cb)]$ by (2). Adding $[w(cb)]g$ to both sides we have $2[w(cb)]g=(cb)(wg)-(gw)(cb)+2g\circ[w(cb)]$ and similarly $2g[w(cb)]=(gw)(cb)-(cb)(wg)+2g\circ[w(cb)]$. But $g\circ[w(cb)]$ is in $\mathfrak{N}$ [**4**, Lemma 14]. Therefore $2[w(cb)]g$ and $2g[w(cb)]$ are in $\mathfrak{M}$ since cb is in $\mathfrak{N}$ and all multiples of elements of $\mathfrak{N}$ are in $\mathfrak{M}$. Similarly $(bh)g=[(bc)w]g$ and $g(bh)=g[(bc)w]$ are in $\mathfrak{M}$. We also have from Lemma 3.4 that $4[h(bz)]g=4[(wc)(bz)]g=2[(bc+cb)\circ wz]g+4(wc\circ bz)g$. Using the elements bz, wc, g in (2) we get $[(bz)(wc)]g+[g(wc)](bz)=(bz)[(wc)g]+g[(wc)(bz)]$. If we add $[(wc)(bz)]g$ to both sides we obtain $2[(wc)\circ(bz)]g=(bz)[(wc)g]-[g(wc)](bz)+2g\circ[(wc)(bz)]$. A similar application of (2) with the elements $2b\circ c$, wz and g yields $2[(bc+cb)\circ wz]g=(bc+cb)[(wz)g]-[g(wz)](bc+cb)+2g\circ[(wz)(bc+cb)]$. Therefore we have $4[h(bz)]g=(bc+cb)[(wz)g]-[g(wz)](bc+cb)+2g\circ[(wz)(bc+cb)]+2(bz)[(wc)g]-2[g(wc)](bz)+4g\circ[(wc)(bz)]$. Since $b\in\mathfrak{N}$ it is clear that $(bc+cb)[(wz)g]$, $[g(wz)](bc+cb)$, $2(bz)[(wc)g]$ and $2[g(wc)](bz)$ are in $\mathfrak{M}$. Both $(wc)(bz)$ and $(wz)(bc+cb)$ are in $\mathfrak{M}$ by definition. Therefore $g\circ[(wz)(bc+cb)]$ and $g\circ[(wc)(bz)]$ are in $\mathfrak{M}$ [**4**, Lemma 17]. Hence $[h(bz)]g$ is in $\mathfrak{M}$. We may now conclude that $(ha)g$ is in $\mathfrak{M}$ for every $h\in\mathfrak{A}_u(1)$, $g\in\mathfrak{G}$ and $a\in\mathfrak{N}$. Similarly $(ah)g$, $g(ah)$ and $g(ha)$ are in $\mathfrak{M}$.

We finally consider the products $(ha)(wd)$ for $d\in\mathfrak{B}$. We have $(hb)(wd)=[w(cb)](wd)=(cb)d$ which is in $\mathfrak{N}$ if $b\in\mathfrak{N}$. Now $2[h(bz)](wd)=2[(wc)(bz)](wd)=[(bc+cb)\circ(wz)](wd)+2(wc\circ bz)(wd)=g(wd)$ where $g=(bc+cb)\circ wz+2(wc\circ bz)$ is an element of $\mathfrak{G}$ by Lemma 3.4. If we use (2) with g, w and d we have $g(wd)+d(wg)=(gw)d+(dw)g$. But w commutes with d and anti-commutes with g. Therefore we have $2[h(bz)](wd)=gw\circ d+(dw)\circ g$. Using (2) with the elements (bz), wc and w we have $[(bz)(wc)]w+c(bz)=(bz)c+w[(wc)(bz)]$. Since $w\circ[(wc)(bz)]=0$ by Lemma 3.4 and the definition of $\mathfrak{G}$ this relation reduces to $2[(bz)\circ(wc)]w=(bc-cb)z\in\mathfrak{M}$. Using (2) with the elements $(bc+cb)$, wz and w we get $2[(bc+cb) \circ wz]w = (bc+cb)[(wz)w] - [w(wz)](bc+cb) + 2w \circ [(wz)(bc+cb)]$. Since $bc+cb$ is in $\mathfrak{N}$ the right side of this identity is in $\mathfrak{M}$. Hence $gw \circ d$ is in $\mathfrak{M}$. We have $(dw) \circ g \in \mathfrak{M}$ [**4**, Lemma 14]. Therefore $[h(bz)](wd)$ is in $\mathfrak{M}$ and $(ha)(wd)$ is in $\mathfrak{M}$ for all $a \in \mathfrak{N}$ and all $d \in \mathfrak{B}$. Every $h' \in \mathfrak{A}_n(1)$ can be expressed as $h' = wd + g$ where $d \in \mathfrak{B}$ and $g \in \mathfrak{G}$. Therefore we have proved that $(ha)h' \in \mathfrak{M}$. In a similar manner we can show that $h'(ha)$, $h'(ah)$ and $(ah)h'$ are in $\mathfrak{M}$. Therefore $\mathfrak{J}\mathfrak{A}_u(1) + \mathfrak{A}_u(1)\mathfrak{J} \subseteq \mathfrak{M}$.

Since $\mathfrak{A}$ is simple and $\mathfrak{M}$ is an ideal of $\mathfrak{A}$ we must have that $\mathfrak{M} = \mathfrak{A}$ or $\mathfrak{M} = 0$. But $1 \notin \mathfrak{N}$, therefore $\mathfrak{M} = 0$ and $\mathfrak{N} = 0$. From this we may conclude that $\mathfrak{A}^+$ is a Jordan algebra [**4**, p. 331]. Hence $\mathfrak{A}^+$ is simple [**5**, Theorem 4.1] and $\mathfrak{A}$ is a flexible, J-simple algebra [**2**, Chapter V, §4]. Therefore we have

THEOREM 3.6. *Every simple, flexible, power-associative, stable algebra of degree two over an algebraically closed field $\mathfrak{F}$ of characteristic $\neq 2, 3, 5$ is a J-simple algebra.*

BIBLIOGRAPHY

1. A. A. Albert, *On Jordan algebras of linear transformations*, Trans. Amer. Math. Soc. vol. 59 (1946) pp. 524–555.

2. ———, *Power-associative rings*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 552–593.

3. ———, *A theory of power-associative commutative algebras*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 503–527.

4. ———, *On commutative power-associative algebras of degree two*. Trans. Amer. Math. Soc. vol. 74 (1953) pp. 323–343.

5. R. H. Oehmke, *On flexible algebras*, Ann. of Math. vol. 68 (1958) pp. 221–230.

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