ON FLEXIBLE POWER-ASSOCIATIVE ALGEBRAS OF DEGREE TWO

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Let \( \mathcal{A} \) be a simple, flexible, power-associative algebra of degree two over an algebraically closed field \( \mathbb{F} \) of characteristic not equal to 2, 3 or 5. Then \( \mathcal{A} \) has a unity element [5, Theorem 3.5] \( 1 = u + v \) where \( u \) and \( v \) are absolutely primitive orthogonal idempotents of \( \mathcal{A} \). The algebra \( \mathcal{A} \) can be decomposed as a vector space direct sum \( \mathcal{A} = \mathcal{A}_u(1) + \mathcal{A}_u(0) \) where \( x \) is in \( \mathcal{A}_u(\lambda) \) if and only if \( ux + xu = \lambda x \). If \( \mathcal{A}_u(\lambda) \mathcal{A}_u(1) + \mathcal{A}_u(1) \mathcal{A}_u(\lambda) \subseteq \mathcal{A}_u(1) \) for \( \lambda = 2, 0 \), \( u \) is called a stable idempotent and \( \mathcal{A} \) is said to be \( u \)-stable. If \( \mathcal{A} \) is \( u \)-stable for every idempotent \( u \) in \( \mathcal{A} \) then \( \mathcal{A} \) is called a stable algebra.

In §1 it is shown that if \( \mathcal{A} \) is a \( u \)-stable algebra then there is an element \( w \) in \( \mathcal{A}_u(1) \) such that \( w^2 = 1 \). The existence of this element is used in §2 to develop some of the multiplicative properties of \( \mathcal{A} \). Finally in §3 it is proved that every simple, flexible, stable, power-associative algebra of degree two over an algebraically closed field \( \mathbb{F} \) of characteristic \( \neq 2, 3, 5 \) is a noncommutative Jordan algebra.

1. Let \( \mathcal{A} \) be a simple, flexible, power-associative algebra of degree two over an algebraically closed field \( \mathbb{F} \) of characteristic \( \neq 2, 3, 5 \). We have the linearization of the power-associative identity

\[
\sum (xy + yx)(zw + wz) = \sum [(xy + yx)z]w \text{ (symmetric in } x, y, z \text{ and } w)\]

and the flexible identity

\[
(xy)z + (zy)x = x(yz) + z(xy).
\]

The algebra \( \mathcal{A} \) has an attached algebra \( \mathcal{A}^+ \) which is the same vector space as \( \mathcal{A} \) but has a product \( x \circ y \) defined by \( x \circ y = (1/2)(xy + yx) \) where \( xy \) is the product of \( \mathcal{A} \). The idempotent \( u \) of \( \mathcal{A} \) is an idempotent of \( \mathcal{A}^+ \) and \( \mathcal{A}_u(\lambda) = \mathcal{A}_u^+(\lambda) \). Since \( u \) and \( v \) are absolutely primitive \( \mathcal{A}_u^+(2) = u \mathcal{A}_2 + \mathcal{N}_2 \) and \( \mathcal{A}_u^+(0) = v \mathcal{A}_0 + \mathcal{N}_0 \) where \( \mathcal{N}_2 \) and \( \mathcal{N}_0 \) are subalgebras of nilpotent elements of \( \mathcal{A}^+ \) [4, Proof of Theorem 3]. The notations \( \mathcal{N} \) shall be used in the sum \( \mathcal{N}_2 + \mathcal{N}_0 \) of the two algebras \( \mathcal{N}_2 \) and \( \mathcal{N}_0 \);

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2 Since Albert's result [2, Theorem 5, page 562] on the idempotent decomposition of a flexible ring is basis to all of our results no further reference shall be made to it each time it is used.
Lemma 1.1. If \( x \in \mathfrak{A}_u(2) \) and \( y \in \mathfrak{A}_u(1) \) then \((xy)_1 = 2\left[u(x o y)\right]_1\) = \(2(x o uy)1\) and \((yx)_1 = 2(x o yu)1 = 2\left[(ux o y)u\right]_1\). (We use the symbol \( x_\lambda \) to denote the \( \mathfrak{A}_u(\lambda) \) component of the element \( x \).)

Proof. By the flexible identity (2) we have \( (xw)u + (uw)x = x(wu) + u(wx)u \). Adding these identities we get \( 2u \circ xw + wu x = xw + x(wu) + 2u \circ wx \). But \( 2(u \circ xw)_1 = (xw)_1 \) and \( 2(u \circ wx)_1 = (wx)_1 \). Therefore \( [(uw)x]_1 = [x(wu)]_1 \). Now \( wu = w - uw \). Therefore \( [(uw)x]_1 = (xw)_1 - [x(wu)]_1 \) and \( 2(x \circ wu)_1 = (xw)_1 \). Similarly \( 2(x \circ wu)_1 = (wx)_1 \). We also have \( [(wx)_1 = [u(wx)]_1 \). Adding \([u(wx)]_1 \) to both sides gives us \( (xw)_1 = 2[u(xo w)]_1 \).

If \( \mathfrak{N}_2 \) is not a subalgebra of \( \mathfrak{A}_u(2) \) we can find two elements \( x \) and \( y \) in \( \mathfrak{N}_2 \) such that \( xy = u + n \) and \( yx = -u + n' \) with \( n \) and \( n' \) in \( \mathfrak{N}_2 \). Let \( a \) be an arbitrary element of \( \mathfrak{A}_u(1) \). From (2) we have \( ua + na + (ay)x = x(ya) + an' - au \) and \( a = x(ya) + an' - (ay)x - na \). Together with Lemma 1.1 this gives us \( a = 4[x o u(y o ua)]_1 + 2[n' o au]_1 - 4[x o (y o au)u]_1 - 2[n o ua]_1 \). Now each element \( x \) of \( \mathfrak{A}_u(2) \) corresponds to a linear transformation \( S_x \) on \( \mathfrak{A}_u(1) \) to \( \mathfrak{A}_u(1) \) such that \( S_x: w \rightarrow (x o w)_1 \). \( S_x \) is nilpotent when \( x \) is nilpotent [3, Lemma 1]. The associative enveloping algebra of the linear transformations \( S_x \) is nilpotent [1, Theorem 8]. We can now write \( a = 2[u(ua)]S_uS_x + 2(au)S_uS_x - 2[(au)u]S_xS_u - 2(ua)S_uS_x \). Since \( a \) is an arbitrary element of \( \mathfrak{A}_u(1) \), a similar expression can be obtained for the elements \( u(ua), au, (au)u \) and \( ua \) of \( \mathfrak{A}_u(1) \). Therefore by an inductive argument we have that \( a \) is a sum of terms of the form \( wS_xS_z \cdots S_z \), where \( w \in \mathfrak{A}_u(1), x_i \in \mathfrak{N}_2 \) and \( t \) is arbitrary. But \( t \) can be chosen large enough so that all products \( S_z \cdots S_z = 0 \). Therefore \( a = 0 \) and \( \mathfrak{A}_u(1) = 0 \). But then \( \mathfrak{A} = \mathfrak{A}_u(2) \oplus \mathfrak{A}_u(0) \) contradicting the assumption that \( \mathfrak{A} \) is simple. Therefore we have

Theorem 1.2. \( \mathfrak{A}_u(2) = u\mathfrak{N} + \mathfrak{N}_2 \) where \( \mathfrak{N}_2 \) is a subalgebra in \( \mathfrak{A} \) of nilpotent elements of \( \mathfrak{A}_u(2) \).

Our algebra shall be restricted further by imposing the condition that \( u \) be a stable idempotent; i.e., \( \mathfrak{A} \) is \( u \)-stable. With this added assumption we prove the existence of an element \( w \) in \( \mathfrak{A}_u(1) \) such that \( w^2 = 1 \). We need the following lemmas.

\[ \text{To avoid excessive use of parentheses we shall use the convention that the multiplicative operation in } \mathfrak{A} \text{ takes precedence over the multiplicative operation of } \mathfrak{A}_u. \]
Lemma 1.3. If $s$ and $t$ are in $A_u(1)$ then $(st)_2 = 2(us \circ t)_2$ and $(ts)_0 = 2(us \circ t)_0$.

Proof. By the flexible identity (2) we have $t(su) + u(st) = (ts)u + (us)t$ and $ts - (us)t + u(st) = (ts)u + (us)t$. Therefore $ts + u(st) = (ts)u + 2(us) \circ t$. Equating $A_u(2)$ components we get $(st)_2 = 2(us \circ t)_2$. Equating $A_u(0)$ components we get $(ts)_0 = 2(us \circ t)_0$.

Lemma 1.4. The set $\mathcal{I}_2$ of all elements of $A_u(2)$ of the form $\sum (ts)_2$ is an ideal of $A_u(2)$ and the set $\mathcal{I}_0$ of all elements of $A_u(0)$ of the form $\sum (ts)_0$ is an ideal of $A_u(0)$ for $t$ and $s$ in $A_u(1)$.

Proof. Let $y = v, x \in A_u(2), w = t$ and $z = s$ be elements of $A_u(1)$ in (1) to get $(xt)s + (tx)s + s(1)x + t(sx) + (xt)x + (tx)x + t(x) = [(xt + tx)s]v + [(xt + tx)v]s + [(xs + sx)t]v + [(xs + sx)v]t + (tx)s + (ts)x + (st)x + (sx)t$. The $A_u(2)$ component of each of the above terms, except possibly $(ts + st)x$, is clearly in $\mathcal{I}_2$. Therefore, by equating components, the $A_u(2)$ component of $(ts + st)x$ is also in $\mathcal{I}_2$. Since $t$ and $s$ are arbitrary elements of $A_u(1), (ut \circ s)_2x$ is in $\mathcal{I}_2$ and by Lemma 1.3, $2(u \circ s)_2x = (ts)_2x = [(ts)x]_2$ is in $\mathcal{I}_2$. Now $x(ts) + s(tx) = (xt)s + (st)x$. Therefore $[(x(ts)]_2$ is also in $\mathcal{I}_2$. We show the second part of the lemma in a similar manner.

Now assume that for every $s, t$ of $A_u(1)$ we have $st \in \mathcal{I}_2 + \mathcal{I}_0 + A_u(1)$. Then $ts \in \mathcal{I}_2 + \mathcal{I}_0 + A_u(1)$ and $t \circ s \in \mathcal{I}_2 + \mathcal{I}_0$. Since $A$ is $u$-stable we have that $\mathcal{I}_2 + A_u(1) + \mathcal{I}_0$ is an ideal of $A$ for $\mathcal{I}_2$ and $\mathcal{I}_0$ defined as in Lemma 1.4. But this ideal of $A$ is contained in $\mathcal{I}_2 + A_u(1) + \mathcal{I}_0$. Therefore the ideal must be zero since $A$ is simple. But this implies that $A_u(1) = 0$ which contradicts the simplicity of $A$. We can conclude that there is a pair of elements $s$ and $t$ in $A_u(1)$ such that $st \in \mathcal{I}_2 + A_u(1) + \mathcal{I}_0$. This implies that either $(st)_2$ is not in $\mathcal{I}_2$ or $(st)_0$ is not in $\mathcal{I}_0$. By Lemma 1.3 we then have that either $(t \circ us)_2 \in \mathcal{I}_2$ or $(ut \circ s)_0 \in \mathcal{I}_0$ and therefore either $t \circ us$ or $ut \circ s$ is not in $\mathcal{I}_2 + A_u(1) + \mathcal{I}_0$. Hence there is a pair of elements $s$ and $t$ in $A_u(1)$ such that $s \circ t \in \mathcal{I}_2 + \mathcal{I}_0$. Since $2s \circ t = (s+t)^2 - s^2 - t^2$, there is an element $y$ in $A_u(1)$ such that $y^2$ is nonsingular. We can now state Albert's result [4, Lemma 3] as

Theorem 1.5. If $y$ is a nonsingular element of $A_u(1)$ then there exists an $x$ in $\mathcal{F}(y)$, the algebra generated by $y$ over $\mathcal{F}$, such that $w = x \circ y$ is in $A_u(1)$ and $w^2 = 1$.

From the existence of such an element $w$ in $A_u^+(1)$ Albert has shown [4] that $A^+$ has the following properties.

Lemma 1.6. Let $B$ be the set of all elements $b$ of $C$ such that $(b \circ w) \circ w = b$. Then $B^+$ is a subalgebra of the algebra $C^+$; both $A_u^+(2)$ and $A_u^+(0)$
are isomorphic to \( \mathcal{B}^+ \). \( \mathcal{A}_u(2) = u \mathcal{B}, \mathcal{A}_u(0) = v \mathcal{B}, \mathcal{C} = \mathcal{B} + \mathcal{B}z, \) and \( w \circ (c \circ w) \) is in \( \mathcal{B} \) for every \( c \) in \( \mathcal{C} \).

**Lemma 1.7.** Let \( a \) and \( b \) be in \( \mathcal{B} \). Then \( (w \circ a) \circ b = (w \circ b) \circ a = w \circ (a \circ b), \ (w \circ a) \circ (w \circ b) = a \circ b, \ w \circ \mathcal{B}z = 0 \) and \( w \circ (a \circ u) = w \circ (a \circ v) = (w \circ a)/2. \)

**Lemma 1.8.** \( \mathcal{A}_u(1) = w \circ \mathcal{B} + \mathcal{G} \) where \( \mathcal{G} \) consists of all elements \( g \) of \( \mathcal{A}_u(1) \) such that \( w \circ g = 0 \) and if \( g \in \mathcal{G}, b \in \mathcal{B} \) then there exists a \( d \in \mathcal{B} \) such that \( g \circ (z \circ b) = w \circ d. \)

**Lemma 1.9.** The element \( e = (1 + w)/2 \) is an idempotent of \( \mathcal{A} \) and \( \mathcal{A}_e(2) = e \mathcal{B}, \mathcal{A}_e(1) = e \mathcal{B} + \mathcal{G}. \)

2. In this section we shall develop multiplicative properties for \( A \) similar to those that Albert has developed for \( \mathcal{A}^+. \)

**Lemma 2.1.** For every \( c \) in \( \mathcal{C} \) we have \( w(w \circ c) = (w \circ c)w \in \mathcal{B}. \)

**Proof.** From (2) we have \( w(wc) +cw^2 = wc^2 + (cw)w \) and \( w(cw) = (wc)w. \) Adding these two identities and using the fact that \( w^2 = 1 \) we have \( w(c \circ w) = (c \circ w)w = w \circ (c \circ w). \) Hence the lemma follows from Lemma 1.6.

**Lemma 2.2.** If \( x \in \mathcal{A}_u(2) \) and \( w \in \mathcal{A}_u(1) \) then \( u(wx) = (xw)u. \) If \( x \in \mathcal{A}_u(0) \) and \( w \in \mathcal{A}_u(1) \) then \( u(xw) = (wx)u. \)

**Proof.** By the flexible identity (2) with \( x \in \mathcal{A}_u(2) \) we have \( u(xw) + wx = xw + (wx)u. \) But \( u(xw) = xw - (xw)u \) and \( (wx)u = wx - u(wx). \) Therefore \( (xw)u = u(wx). \) The second result of the lemma is obtained in a similar manner.

**Lemma 2.3.** If \( c \in \mathcal{A}_u(2) \) then \( 2[(c \circ w)w] = c. \) If \( b \in \mathcal{B} \) then \( 2[(b_0 \circ w)w] = b_2 = 2[(b_2 \circ w)w]. \)

**Proof.** Let \( c \) be an element of \( \mathcal{C}. \) Let \( x = c, y = u \) and \( z = w \) in (1). Using \( w^2 = 1 \) and Lemma 2.1 we have \( 4c_2 + 8(c \circ w)w = 4(c_2w)w + 2(wc)w + 2c + 4[(c \circ w)u]w + 4[(c \circ w)w]u. \) From Lemma 2.2 we have \( 2(c \circ w)u = wc_2 + c_0w. \) Therefore the above identity can be simplified to \( 4c_2 + 8(c \circ w)w = 8(c_2 \circ w)w + 4(w \circ c_0w)w + 2c + 4[(c \circ w)w]u. \) If \( c = c_2 \) then \( c_0 = 0 \) and \( 2c_2 = 4[(c_2 \circ w)w]. \) If \( c \in \mathcal{B} \) then \( (c \circ w)w = c \) and \( 4(c_0 \circ w)w = 2c. \) Therefore \( 4(c_2 \circ w)w = 2c. \) Equating \( \mathcal{A}_u(2) \) components we have \( 2[(c_0 \circ w)w] = 2[(c_2 \circ w)w] = c_2. \)

**Lemma 2.4.** If \( a \) and \( b \) are elements of \( \mathcal{B} \) then \( wb = bw = b \circ w \) and \( b(aw) = (ba) \circ w \) and \( (wa)b = w \circ ab. \)
Proof. Let \( x = y = w \) and \( z = b \) in (1) to get \( 4(w \circ b) = wb + bw + [(bw + wb)w]w \) and \( (w \circ b) = [(b \circ w)w]w \). But by Lemma 2.1 and the definition of \( \mathcal{B} \) we have \( [(b \circ w)w]w = bw \). Therefore \( bw = wb = b \circ w \). This implies that \( 2b \circ (a \circ w) = b(aw) - (aw)b \). Since \( \mathcal{B}^+ \) is a subalgebra of \( A^+ \) we also have that \( 2(a \circ b) \circ w = 2(a \circ b)w = 2w(a \circ b) \). But \( 2b \circ (a \circ w) = 2(a \circ b) \circ w \) from Lemma 1.7. Combining these results with \( (wa)b + (ba)w = w(ab) + b(aw) \) obtained from (2) we have \( b(aw) = w \circ ba \) and \( (wa)b = w \circ ab \).

Theorem 2.5. \( \mathcal{B} \) is a subalgebra of \( \mathcal{A} \).

Proof. Using the elements \( wa, w \) and \( b \) in (2) we have \( (wa)(wb) + b[w(wa)] = [(wa)w]b + (bw)(wa) \) and \( (wa)(wb) + ba = ab + (bw)(wa) \). From Lemma 1.7 and Lemma 2.4 we have \( (wa)(wb) = -(bw)(wa) + ab + ba \). Combining these results we get \( (wa)(wb) = ab \) and \( (wb)(wa) = ba \). Since \( e = (1 + w)/2 \) is an idempotent of \( \mathcal{A}^+ \) it is also an idempotent of \( \mathcal{A} \) and \( \mathcal{A}_e(2) \) is a subalgebra of \( \mathcal{A} \). From Lemma 1.9 the subspace \( \mathcal{A}_e(2) \) is equal to \( e\mathcal{B} \). Therefore every element of \( \mathcal{A}_e(2) \) can be written in the form \( b + bw \) where \( b \in \mathcal{B} \). If \( a \) and \( b \in \mathcal{B} \) then there exists \( d \in \mathcal{B} \) such that \( (a + wa)(b + wb) = d + wd \). Multiplying out the left side and equating components in the decomposition of \( \mathcal{A} \) with respect to the idempotent \( u \) we have \( 2ab = d \). Therefore \( ab \in \mathcal{B} \). Hence \( \mathcal{B} \) is closed under the multiplication of \( \mathcal{A} \). The other properties of an algebra obviously hold in \( \mathcal{B} \).

It is easily seen from the above lemmas, in particular Lemma 2.3, that the mapping \( b \rightarrow bu \) is an isomorphism of \( \mathcal{B} \) onto \( \mathcal{A}_e(2) \). Similarly \( \mathcal{B} \) and \( \mathcal{A}_e(0) \) are isomorphic. Therefore we have

Theorem 2.6. \( \mathcal{A}_e(2) \) and \( \mathcal{A}_e(0) \) are isomorphic subalgebras of \( \mathcal{A} \).

3. In addition to our previous assumptions on \( \mathcal{A} \) we shall now add the assumption that \( A \) is stable. We proceed with a sequence of lemmas leading to our main theorem that \( A \) is a noncommutative Jordan algebra.

Lemma 3.1. If \( g \in \mathcal{G} \) and \( b \in \mathcal{B} \) then \( gb \) and \( bg \in \mathcal{G} \).

Proof. Since \( g \in \mathcal{A}_e(1) \) and \( b \in \mathcal{A}_e(2) + \mathcal{A}_e(0) \) we have \( gb \in \mathcal{A}_e(1) \). We can write \( gb = wd + h \) where \( d \in \mathcal{B} \) and \( h \in \mathcal{G} \) by Lemma 1.8. But we also have \( g \in \mathcal{A}_e(1) \) and \( b \in \mathcal{A}_e(2) + \mathcal{A}_e(0) \). Therefore \( gb \in \mathcal{A}_e(1) \) and we can write \( gb = za + h' \) where \( a \in \mathcal{B} \) and \( h' \in \mathcal{G} \). Equating components of the two representations of \( gb \) we have \( h = h' \), \( wd = 0 \) and \( za = 0 \). Hence \( gb \in \mathcal{G} \). In a similar manner we show that \( bg \in \mathcal{G} \).

Lemma 3.2. For all \( d \in \mathcal{B} \) we have \( (wd)z = d \circ wz = w(dz) \).
PROOF. From (2) we have \((zw)d + (dw)z = z(wd) + d(wz)\). Since \(d\) commutes with \(w\) and \(z\) anti-commutes with all elements of \(\mathfrak{A}_n(1)\) we have \(2(dw)z = d(wz) - (wz)d = 2d \circ wz\). Also \((wd)z + (zd)w = w(dz) + z(dw)\). Since \(w \circ dz = 0\) we have \(w(dz) = -(dz)w\) and \((zd)w = z(dw)\).

**Lemma 3.3.** For every \(a\) and \(b \in \mathfrak{B}\) we have \(2[(aw)(bz)]z = (ab)w - (ba)w + [(wz)z] \circ d\) where \(d = 2a \circ b\).

**Proof.** Let \(x = bz\) and \(y = a\) in (1) to obtain \(iwd = dw + [(dz)w]z + 2(ba)w + 2(bw)a + 2[(aw)z](bz) + 2[(aw)(bz)]z + 2[(aw)(bz)]w + 2[(az)(bz)]w\). But from Lemmas 1.7 and 3.2 we have \(- (az)w = w(az) = (wa)z = (aw)z\). Therefore the above identity can be reduced to \(4wd = 3(ab)w + 5(ba)w + [(wz)z]z + 2[(aw)(bz)]z\). From Lemma 3.2 we have \([dz)w]z = -(w(dz)z = -(d \circ wz)z\). By (2) we have \([d(wz)]z = -(wz)z]z + d[(wz)z]z\). Therefore \(2(d \circ wz)z = [d(wz)]z + d[(wz)z]z\). Hence \([dz)w]z = -(d \circ [(wz)z]z\) and \(2[(aw)(bz)]z = w(ab) - w(ba) - [(wz)z] \circ d\).

**Lemma 3.4.** For every pair of elements \(d\) and \(b\) in \(\mathfrak{B}\) we have \(2(wd)(bz) = (bd + db) \circ wz \circ 2wd \circ bz \in \mathfrak{G}\).

**Proof.** From (2) we have \((wd)(bz) + z[(bd)w] = [(bd)w]z + (sb)(zd)\). Adding \((wd)(bz)\) to both sides and using Lemma 3.2 we have \(2(wd)(bz) = [w(bd)]z + [w(db)]z + 2(sb) \circ (wd) = (bd + db) \circ wz + 2(sb) \circ (wd)\). We can apply [4, Lemma 10] to get \((sb) \circ (wd)\) in \(\mathfrak{G}\). To show that \((bd + db) \circ wz\) is in \(\mathfrak{G}\) we substitute \(w, w\) and \(z\) in (2) to get \(w(wz) + zw^2 = wz + (zw)w\). Since \(z\) anti-commutes with \(w\) we have \(w \circ wz = 0\). Therefore \(wz \in \mathfrak{G}\) and by Lemma 3.1 \((bd + db) \circ wz \in \mathfrak{G}\).

**Lemma 3.5.** The set \(\mathcal{M} = \mathfrak{N} + \mathfrak{A}_n(1)\mathfrak{N} + \mathfrak{A}_n(1)\) is an ideal of \(\mathfrak{A}\).

**Proof.** Since \(\mathfrak{N}\) is an ideal of \(\mathfrak{C}\) we clearly have that \(\mathfrak{M} + \mathfrak{N} \subseteq \mathfrak{M}\). Therefore it is sufficient to show that \(\mathfrak{N} \circ \mathfrak{M} \subseteq \mathfrak{M}\) where \(\mathfrak{J} = \mathfrak{A}_n(1)\mathfrak{N} + \mathfrak{A}_n(1)\subseteq \mathfrak{A}_n(1)\). If \(h\) is any element of \(\mathfrak{A}_n(1)\) we can find an \(\alpha\) in \(\mathfrak{F}\) such that \(h + \alpha w\) is nonsingular. Therefore every \(h\) in \(\mathfrak{A}_n(1)\) can be expressed as the difference of two nonsingular elements of \(\mathfrak{A}_n(1)\). Hence it will be sufficient to show that all multiples of \(ha\) and \(ah\) are in \(\mathfrak{M}\) for \(h\) in \(\mathfrak{A}_n(1)\) and nonsingular and for \(a\) in \(\mathfrak{N}\). Let \(h\) be any nonsingular element of \(\mathfrak{A}_n(1)\) then by Theorem 1.5 there is an \(a\) in \(\mathfrak{F}(h) \cap \mathfrak{C}\) such that \(w = h \circ d\) is in \(\mathfrak{A}_n(1)\) and \(w^2 = 1\). Since the associative and commutative laws hold in \(\mathfrak{F}(h)\) and \(d\) is nonsingular there is a \(c \in \mathfrak{F}(h) \cap \mathfrak{C}\) such that \(w \circ c = h\) and \((c \circ w) \circ w = c\). Therefore \(c\) is in the algebra \(\mathfrak{B}\) corresponding to this \(w\). Similarly, \(d \in \mathfrak{B}\). Now let \(b \in \mathfrak{B}\) then \((h)z = [(wc)z]z = [w \circ cb]z = wzc \circ cb\) by Lemmas 2.4 and 3.2.
Therefore if $b$ is nilpotent $(hb)z$ is in $\mathcal{Z}$. Also $2[h(bz)]z = 2[(wc)(bz)]z = (cb)w - (bc)w + [(wc)z] o (cb + bc)$ by Lemma 3.3. Again if $b$ is nilpotent $(b(bz)]z$ is in $\mathcal{Z}$. But every element $a$ of $\mathcal{A}$ can be expressed in the form $b + b'z$ where $b$ and $b'$ are nilpotent elements of $\mathcal{A}$. Therefore we have that $(ha)z$ is in $\mathcal{Z}$ for all nilpotent elements $a$ in $\mathcal{C}$ and all $h$ in $\mathcal{A}_u(1)$. A similar argument is used to show that $(ah)z$ is in $\mathcal{Z}$. Hence $z^3 = -3z^2 \mathcal{Z}$. Let $x$ be any element of $\mathcal{C}$. We can express $x$ as $x = \alpha z + \beta 1 + n$ where $\alpha$ and $\beta$ are in $\mathcal{Z}$ and $n \in \mathcal{A}$. Since $\mathcal{Z} \subseteq \mathcal{A}_u(1)$ it is clear that $\mathcal{Z} + \mathcal{Z} \subseteq \mathcal{Z}$. Therefore we may conclude that for any $x \in \mathcal{C}$ we have $x^3 \subseteq \mathcal{Z}$.

It remains to show that $\mathcal{A}_u(1) \mathcal{Z} + \mathcal{Z} \mathcal{A}_u(1) \subseteq \mathcal{M}$. Let $g \in \mathcal{G}$ and $hb$ as above. We have $(hb)z = [(wc)b]g = [w(cb)]g = [(cb)w]g = -(gw)(cb) + (cb)(wg) - g[w(cb)]$ by (2). Adding $[w(cb)]g$ to both sides we have $2[w(cb)]g = (cb)(wg) - (gw)(cb) + 2g o [w(cb)]$ and similarly $2g[w(cb)] = (gw)(cb) - (cb)(wg) + 2g o [w(cb)]$. But $g o [w(cb)]$ is in $\mathcal{M}$ [4, Lemma 14]. Therefore $2[w(cb)]g$ and $2g[w(cb)]$ are in $\mathcal{M}$ since $cb$ is in $\mathcal{M}$ and all multiples of elements of $\mathcal{M}$ are in $\mathcal{M}$. Similarly $(bh)g = [(bc)w]g$ and $(bh)g = g[(bc)w]g$ are in $\mathcal{M}$. We also have from Lemma 3.4 that $4[h(bz)]g = 4[(wc)(bz)]g = 2[(bc + cb) o wz]g + 4(wc o bz)g$. Using the elements $b, wc, g$ in (2) we get $[(bc)w]g + g[(wc)](bz) = (bc)[(wc)g] + g[(wc)](bz)$. If we add $[(wc)(bz)]g$ to both sides we obtain $2[(wc) o (bz)]g = (bc)[(wc)g] - [g(wc)](bz) + 2g o [(wc)](bz)$. A similar application of (2) with the elements $b o c$, $wz$ and $g$ yields $2[(bc + cb) o wz]g = (bc + cb)[(wz)g] - [g(wz)](bc + cb) + 2g o [(wz)(bc + cb)]$. Therefore we have $4[h(bz)]g = (bc + cb)[(wz)g] - [g(wz)](bc + cb) + 2g o [(wz)(bc + cb)] + 2[(wz)]g - 2[g(wc)](bz) + 4g o [(wc)](bz)$. Since $b \in \mathcal{M}$ it is clear that $(bc + cb)[(wz)g]$, $[g(wz)](bc + cb)$, $2[(wz)]g$ and $2[g(wc)](bz)$ are in $\mathcal{M}$. Both $(wc)(bz)$ and $(wz)(bc + cb)$ are in $\mathcal{M}$ by definition. Therefore $g o [(wc)(bz)]$ and $g o [(wc)](bz)$ are in $\mathcal{M}$ [4, Lemma 17]. Hence $h(bz)]g$ is in $\mathcal{M}$.

We may now conclude that $(ha)g$ is in $\mathcal{M}$ for every $h \in \mathcal{A}_u(1)$, $g \in \mathcal{G}$ and $a \in \mathcal{A}$. Similarly $(ah)g$, $(gh)g$ and $(gh)g$ are in $\mathcal{M}$.

We finally consider the products $(ha)(wd)$ for $d \in \mathcal{A}$. We have $(hb)(wd) = [w(cb)](wd) = (cb)d$ which is in $\mathcal{M}$ if $b \in \mathcal{M}$. Now $2[h(bz)](wd) = 2[(wc)(bz)](wd) = [(bc + cb) o (wz)](wd) + 2(wc o bz)(wd) = g(wd)$ where $g = (bc + cb) o wz + 2(wc o bz)$ is an element of $\mathcal{G}$ by Lemma 3.4. If we use (2) with $g$, $w$ and $d$ we have $g(wd) + d(wg) = (gw)d + (dw)g$. But $w$ commutes with $d$ and anti-commutes with $g$. Therefore we have $2[h(bz)](wd) = gw o d + (dw) o g$. Using (2) with the elements $(bs)$, $(wc)$ and $w$ we have $[(bs)(wc)]w + c(bz) = (bs)c + w[(wc)(bz)]$. Since $w o [(wc)](bz)] = 0$ by Lemma 3.4 and the definition of $\mathcal{G}$ this relation reduces to $2[(bz) o (wc)]w = (bc - cb)z \mathcal{M}$. Using (2) with the ele-
ments \((bc+cb), wz\) and \(w\) we get 
\[
2[(bc+cb)\circ wz]w = (bc+cb)[(wz)w - w(wz)](bc+cb) + 2w \circ [(wz)(bc+cb)].
\]
Since \(bc+cb\) is in \(N\) the right side of this identity is in \(M\). Hence \(gw \circ d\) is in \(M\). We have \((dw) \circ g \in M\) \(\text{[4, Lemma 14]. Therefore } [h(bz)](wd)\) is in \(M\) and \((ha)(wd)\) is in \(M\) for all \(a \in N\) and all \(d \in B\). Every \(h' \in A_n(1)\) can be expressed as \(h' = wd + g\) where \(d \in B\) and \(g \in G\). Therefore we have proved that \((ha)h' \in M\). In a similar manner we can show that \(h'(ha), h'(ah)\) and \((ah)h'\) are in \(M\). Therefore \(A_n(1) + A_n(1)G \subseteq M\).

Since \(A\) is simple and \(M\) is an ideal of \(A\) we must have that \(M = A\) or \(M = 0\). But \(1 \in R\), therefore \(M = 0\) and \(N = 0\). From this we may conclude that \(A^+\) is a Jordan algebra \(\text{[4, p. 331]. Hence } A^+\) is simple \(\text{[5, Theorem 4.1]}\) and \(A\) is a flexible, \(J\)-simple algebra \(\text{[2, Chapter V, \S 4]}\). Therefore we have

**Theorem 3.6.** Every simple, flexible, power-associative, stable algebra of degree two over an algebraically closed field \(\mathbb{F}\) of characteristic \(\neq 2, 3, 5\) is a \(J\)-simple algebra.

**Bibliography**


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