NOTE ON THE NONVANISHING OF $L(1)$

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It is well known that if $\chi(m)$ is a real nonprincipal character (mod $k$), then

$$L(1) = \sum_{1}^{\infty} \frac{\chi(m)}{m} \neq 0,$$

and many proofs have been found. We give a very simple proof when $k=p$ an odd prime, in which case $\chi(m) = (m/p)$, the Legendre symbol. This makes it possible to simplify the proof that if $p|a$, then there are infinitely many primes congruent to $a$ modulo $p$. Write

$$\xi = e^{2\pi i/p}, \quad P = \prod_{r}^{n} \frac{1 - \xi^{r}}{1 - \xi},$$

where $n$ runs through the quadratic nonresidues of $p$ and $r$ runs

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through the quadratic residues. We prove first that $L(1) \neq 0$ if $P \neq 1$. Since

$$\frac{1}{1 - Z} = \exp \left\{ \sum_{m=1}^{\infty} \frac{Z^m}{m} \right\} \quad (|Z| \leq 1, Z \neq 1)$$

we have

$$P = \exp \left\{ \sum_{m=1}^{\infty} \frac{1}{m} \left( \sum_r m r^m - \sum_n n r^m \right) \right\}$$

$$= \exp \left\{ S \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{m}{p} \right) \right\} = \exp \{ SL(1) \},$$

where

$$S = \sum_r r^r - \sum_n n r^n = \sum_{m=1}^{p-1} \left( \frac{m}{p} \right) r^m.$$

Hence $L(1) \neq 0$ if $P \neq 1$. Let $c$ be any fixed positive integer which is a quadratic nonresidue of $p$, e.g., $c = p - 1$ if $p \equiv 3 \pmod{4}$. Then since $n \equiv cr \pmod{p}$, the equation $P = 1$ can be written as

$$\prod_r \left( \frac{1 - e^{cr}}{1 - e^r} \right) = 1.$$

Then the polynomial

$$\prod_r \left( \frac{1 - Z^c}{1 - Z^r} \right) - 1$$

has a zero $\zeta$ which satisfies the irreducible equation $1 + Z + Z^2 + \cdots + Z^{p-1} = 0$. Hence if $Z$ is any variable,

$$\prod_r \left( \frac{1 - Z^c}{1 - Z^r} \right) - 1 = f(Z)(1 + Z + Z^2 + \cdots + Z^{p-1}),$$

where $f(Z)$ is a polynomial in $Z$ with integral coefficients. Put $Z = 1$. Then $c^{(p-1)/2} - 1 \equiv 0 \pmod{p}$, which is a contradiction, in view of Euler's criterion for quadratic residuacity. This finishes the proof.

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