NOTE ON THE NONVANISHING OF $L(1)$

S. CHOWLA AND L. J. MORDELL

It is well known that if $\chi(m)$ is a real nonprincipal character (mod $k$), then

$$L(1) = \sum_{1}^{\infty} \frac{\chi(m)}{m} \neq 0,$$

and many proofs have been found. We give a very simple proof when $k=p$ an odd prime, in which case $\chi(m) = (m/p)$, the Legendre symbol. This makes it possible to simplify the proof that if $p|a$, then there are infinitely many primes congruent to $a$ modulo $p$. Write

$$\zeta = e^{2\pi i/p}, \quad p = \frac{\prod (1 - \zeta^n)}{\prod (1 - \zeta^r)},$$

where $n$ runs through the quadratic nonresidues of $p$ and $r$ runs

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through the quadratic residues. We prove first that \( L(1) \neq 0 \) if \( P \neq 1 \). Since

\[
\frac{1}{1 - Z} = \exp \left\{ \sum_{m=1}^{\infty} \frac{Z^m}{m} \right\} \quad (|Z| \leq 1, Z \neq 1)
\]

we have

\[
P = \exp \left\{ \sum_{m=1}^{\infty} \frac{1}{m} \left( \sum_r \zeta^{rm} - \sum_n \zeta^{nm} \right) \right\}
= \exp \left\{ S \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{m}{p} \right) \right\} = \exp \{ SL(1) \},
\]

where

\[
S = \sum_r \zeta^r - \sum_n \zeta^n = \sum_{m=1}^{p-1} \left( \frac{m}{p} \right) \zeta^m.
\]

Hence \( L(1) \neq 0 \) if \( P \neq 1 \). Let \( c \) be any fixed positive integer which is a quadratic nonresidue of \( p \), e.g., \( c = p - 1 \) if \( p \equiv 3 \pmod{4} \). Then since \( n \equiv cr \pmod{p} \), the equation \( P = 1 \) can be written as

\[
\prod_r \left( \frac{1 - \zeta^{cr}}{1 - \zeta^r} \right) = 1.
\]

Then the polynomial

\[
\prod_r \left( \frac{1 - Z^{cr}}{1 - Z^r} \right) - 1
\]

has a zero \( \zeta \) which satisfies the irreducible equation \( 1 + Z + Z^2 + \cdots + Z^{p-1} = 0 \). Hence if \( Z \) is any variable,

\[
\prod_r \left( \frac{1 - Z^{cr}}{1 - Z^r} \right) - 1 = f(Z)(1 + Z + Z^2 + \cdots + Z^{p-1}),
\]

where \( f(Z) \) is a polynomial in \( Z \) with integral coefficients. Put \( Z = 1 \). Then \( c^{(p-1)/2} - 1 \equiv 0 \pmod{p} \), which is a contradiction, in view of Euler's criterion for quadratic residuacity. This finishes the proof.

University of Colorado