ON HOMOTHETIC MAPPINGS OF RIEMANN SPACES

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The object of this note is to generalize some results obtained in [1; 2] and to give a shorter proof to one of them.

1. Let $V_n$ be a Riemann space with fundamental metric $g_{ij}(x)$. Let $\xi^i(x)$ be a vector field defining a one-parameter Lie group and $L$ the symbol of Lie differentiation based on $\xi^i(x)$. The condition that $\xi^i(x)$ define a motion, an affine collineation, a homothetic transformation or a conformal transformation of $V_n$ is

$$(1.1) \quad Lg_{ij} = \xi_{i,j} + \xi_{j,i} = 0,$$

$$(1.2) \quad L\left\{ \frac{i}{jk} \right\} = \xi_{j,k} + R^i_{jkl} \xi^l = 0,$$

$$(1.3) \quad Lg_{ij} = 2cg_{ij},$$

or

$$(1.4) \quad Lg_{ij} = 2\phi g_{ij},$$

respectively, where $\xi_{i,j}$ is the covariant derivative of $\xi^i$ with respect to the Christoffel symbols $\{^i_{jk}\}$ and $R^i_{jkl}$ the curvature tensor of $V_n$, $c$ and $\phi$ being a constant and a function of $x$ respectively. When $c$ vanishes, a homothetic transformation reduces to a motion. Thus we call a proper homothetic transformation one for which $c \neq 0$.

Since we have the formulas

$$(1.5) \quad L\left\{ \frac{i}{jk} \right\} = \frac{1}{2} g^{ia}[(Lg_{ja})_{,k} + (Lg_{ak})_{,j} - (Lg_{jk})_{,a}],$$

$$(1.6) \quad \left( L\left\{ \frac{i}{jk} \right\} \right)_{,i} - \left( L\left\{ \frac{i}{jl} \right\} \right)_{,k} = LR^i_{jkl},$$

it is easily seen that a motion and a homothetic transformation are both affine collineations and that an affine collineation preserves the curvature tensor.

In [1] one of the present authors proved that in a space of nonzero constant curvature a mapping preserving curvature is a motion. For an Einstein space with nonzero curvature scalar, a mapping preserving Ricci curvature is a motion; for applying the operator $L$ to

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\[ R_{ij} = \alpha g_{ij}, \quad \alpha = \text{constant}, \quad n > 2, \]

we have

\[ 0 = LR_{ij} = \alpha Lg_{ij}, \]

from which, \( \alpha \) being different from zero, we have

\[ Lg_{ij} = 0. \]

Since a homothetic transformation is an affine collineation, we have, as a corollary of this:

**Theorem 1.** In an Einstein space of nonzero curvature scalar, a homothetic transformation is a motion.

This generalizes Theorem 2.2 of [2].

For a homothetic transformation, we have

\[ LR = L(g^{ii}R_{ii}) = (Lg^{ii})R_{ij} \]

\[ = -2c^{ii}R_{ij} = -2cR, \]

where \( R \) is the curvature scalar. Thus if \( R = \text{constant} \neq 0 \), then we have \( c = 0 \), from which we have

**Theorem 2.** In a Riemann space with nonzero constant curvature scalar, a homothetic transformation is a motion.

This generalizes Theorem 1.

For a conformal transformation, we have

\[ \phi_{,i,j} = LC_{ij} \]

where

\[ C_{ij} = -\frac{R_{ij}}{n-2} + \frac{Rg_{ij}}{2(n-1)(n-2)}, \quad n > 2. \]

thus if \( R \) is a constant, we have

\[ \tag{1.7} g^{ii}\phi_{,i,j} = -\frac{R}{n-1}\phi, \]

from which we conclude

**Theorem 3.** If a space with constant curvature scalar admits a proper homothetic transformation, then the curvature scalar vanishes.

In the case where the space is compact and orientable, one of the present authors [3] proved that an affine collineation is a motion and so a homothetic transformation is also a motion.
In a compact orientable Riemann space with constant curvature scalar, substituting (1.7) into a famous integral formula

$$\frac{1}{2} \int g^{ij}(\phi^2)_{,i,j} d\sigma = \int (\phi g^{ij}\phi_{,i,j} + g^{ij}\phi_{,i}\phi_{,j}) d\sigma = 0$$

we find

$$\int \left( \frac{R}{n-1} \phi^2 + g^{ij}\phi_{,i}\phi_{,j} \right) d\sigma = 0.$$  

Thus if $R < 0$, then $\phi = 0$ and if $R = 0$, then $\phi = \text{constant}$. When $\phi = 0$ the conformal transformation is a motion and when $\phi = \text{constant}$, the conformal transformation is a homothetic transformation and is consequently an affine collineation. Thus corresponding to Theorem 2, we have

**Theorem 4.** In a compact orientable Riemann space with constant curvature scalar $< 0$, a conformal transformation is a motion.

2. Assume that the Riemann space $V_n$ admits a group $G_{r+1}$ of homothetic transformations and denote by $\xi_{(\alpha)}^i$ ($\alpha, \beta = 1, 2, \cdots, r, r+1$) generators of the group and by $L_\alpha$ the operators of Lie differentiation with respect to $\xi_{(\alpha)}^i$. Then we have

$$L_\alpha g_{ij} = 2c_\alpha g_{ij},$$

where $c$'s are constants not all zero. Without loss of generality we can assume that

$$c_\alpha = 0 \quad (a = 1, 2, \cdots, r).$$

Now the following relation holds good [4]:

$$L_\alpha L_\beta - L_\beta L_\alpha g_{ij} = c_\alpha c_\beta^\gamma L_\gamma g_{ij},$$

where $c$'s are constants of structure.

Putting $\alpha = a, \beta = r+1$ in (2.3), we find

$$c_{a^{r+1}} = 0$$

by virtue of (2.1) and (2.2). Equation (2.4) proves Theorem 2.1 in [2], that is,

The full group $G_{r+1}$ of homothetic transformations of $V_n$ contains an invariant subgroup $G_r$ of motions and a $G_1$ subgroup of dilations.

3. **Finsler spaces.** In such a space arc length is defined by $ds^2 = F(x, dx)$ homogeneous of degree 2 in $dx$. In such a space covariant
differentiation may be defined in different ways but in any case it is not based on the Christoffel symbols.

As for Riemann spaces a motion is defined by

$$L_{g,ij} = \frac{\partial g_{ij}}{\partial (dx^k)} = 0$$

where $g_{ij,k} = g_{ij}^k \xi^h + g_{ij} g_{ik} \xi^i + g_{ij} g_{ik} \xi^h dx^k = 0$

(3.1)

$$L_{g,ij} = 2\phi g_{ij}$$

defines a homothetic mapping if $\phi$ is a constant and a conformal mapping if it is not a constant ($\phi$ is necessarily independent of $dx$). We showed above that a homothetic mapping in a Riemann space is an affine collineation and the converse, i.e., a conformal mapping which is an affine collineation is homothetic. To prove this for a Finsler space we use the fact that for a conformal mapping

$$LY_{jk} = \delta^i_{j} \phi^i + \phi^i,_{j} - \frac{1}{2} (F_g)^{ih},_{j} \phi^i_h.$$  

(3.3)

Obviously if $\phi = \text{constant} \ LT_{jk} = 0$ which defines an affine collineation. To prove the converse we have

$$\delta^i_{j} \phi^i + \phi^i,_{j} - \frac{1}{2} (F_g)^{ih},_{j} \phi^i_h = 0$$

(3.4)

so that

$$2dx^i \phi^i_h dx^h - F(g^{ih} \phi^i_h) = 0.$$  

(3.5)

Contracting the above with $g_{ij,1}$ we find $F g^{ih} g_{ij,1} \phi^i_h = 0$ and since $g^{ih} g_{ij,1} = -g_{ij} g^{ih}$ so that $(g^{ih} \phi^i_h),_1 = 0$.

Differentiating (3.5) partially with respect to $dx^1$ we obtain

$$\delta^i_{j} \phi^i_h dx^k + \phi^i,_{j} dx^i - g_{ij} \delta^i g^{ih} \phi^i_h = 0$$

and contraction gives $n \phi^i_h dx^k = 0$ and hence $\phi = \text{constant}.$

BIBLIOGRAPHY


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