INNER PRODUCT SPACES AND THE TRI-SPHERICAL INTERSECTION PROPERTY[1]

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1. **Introduction and definitions.** This note adds another criterion to the lengthy list of those properties which characterize the real inner product spaces whose dimension exceeds two. For a concise guide to the history of the development of this subject, the reader is referred to Day [**2**, VII, §3].

1.1. DEFINITION. A subset F of a linear space E is said to be a flat if, for some $x \in E$, the set $F-x$ is a linear subspace of E.

1.2. DEFINITION. Let x_1, x_2 and x_3 be points in a normed linear space E. Then $F(x_1, x_2, x_3)$ denotes the smallest flat containing each of the points x_i.

1.3. DEFINITION. If E is a normed linear space and $x \in E$ and $\rho > 0$, then by $S_\rho(x)$ we mean the set $\{y \in E \mid \|x-y\| \leqq \rho\}$.

1.4. DEFINITION. A normed linear space E is said to have the tri-spherical intersection property if, whenever x_1, x_2 and x_3 are points in E and ρ_1, ρ_2 and ρ_3 are positive numbers with $\bigcap_{i=1}^3 S_{\rho_i}(x_i) \neq \emptyset$, it follows that $\bigcap_{i=1}^3 S_{\rho_i}(x_i) \cap F(x_1, x_2, x_3) \neq \emptyset$.

1.5. REMARK. Every two-dimensional real normed linear space has the tri-spherical intersection property. Every real normed linear space has the bi-spherical intersection property, defined by analogy with 1.4.

2. **Theorems from the literature.** We list here three theorems upon which the proof of our result depends.

2.1. DEFINITION. (Birkhoff. See [**1**, p. 169].) If x and y are elements of a real normed linear space E, then we say y is orthogonal to x, and write $y \perp x$, if $\|y - \alpha x\| \geqq \|y\|$ for each real number α. If J is a subset of E, then we write $J \perp x$ if $y \perp x$ for each $y \in J$.

2.2. THEOREM. (James. See [**5**, Theorem 4].) *A real normed linear space whose dimension exceeds 2 is an inner product space provided that for each hyperplane H of E containing ϕ there is a point x of E for which $x \neq \phi$ and $H \perp x$.*

2.3. THEOREM. (Fréchet. See [**3**, p. 717].) *In order that a real*

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normed linear space be an inner product space, it is sufficient that each of its three-dimensional subspaces be an inner product space.

2.4. THEOREM. (Helly. See [4].) *Let D be an n-dimensional real normed linear space, and let $\mathfrak{F}$ be a collection of compact, convex subsets of D. If every $n+1$ elements of $\mathfrak{F}$ have a point in common, then some point of D lies in every element of $\mathfrak{F}$.*

3. Characterization of inner product space.

3.1. THEOREM. *Let E be a real normed linear space of dimension $\geqq 3$. Then the following assertions are equivalent:*
(a) *E is an inner product space,*
(b) *E has the tri-spherical intersection property.*

PROOF. (a)$\Rightarrow$(b). Let x_1, x_2 and x_3 be points in E, and let $z \in \bigcap_{i=1}^3 S_{\rho_i}(x_i)$ for certain positive numbers ρ_i. The set $F(x_1, x_2, x_3)$, being complete in E, is closed in the completion $\overline{E}$ of E. Clearly we may suppose that $\phi \in F(x_1, x_2, x_3)$. Let P denote the orthogonal projection of $\overline{E}$ onto $F(x_1, x_2, x_3)$. Then either each $x_i = \phi$ or $\|P\| = 1$, and in any event we have $\|x_i - P(z)\| = \|P(x_i - z)\| \leqq \|x_i - z\| \leqq \rho_i$ for each i. Thus $P(z) \in \bigcap_{i=1}^3 S_{\rho_i}(x_i)$.

(b)$\Rightarrow$(a). Let D be a three-dimensional linear subspace of E, and let H be a two-dimensional linear subspace of D. In view of 2.2 and 2.3, we need only show that $H \perp x$ for some $x \in D$ with $x \neq \phi$. Choose any $z \in D$ which is not in H. Define $\mathfrak{F} = \{S_\rho(y) \mid \rho > 0,\ y \in H$ and $\|y - z\| \leqq \rho\}$. Since D inherits the tri-spherical intersection property from E, every three elements of $\mathfrak{F}$ have a point in common which belongs to H. Hence by 2.4 there is a point $w \in H$ with the property that if $y \in H$ and $\|y - z\| \leqq \rho$, then $\|y - w\| \leqq \rho$, i.e., with the property that if $y \in H$, then $\|y - w\| \leqq \|y - z\|$. Now define $x = z - w$. Since $z \notin H$ and $w \in H$, we have $x \neq \phi$. Then for each nonzero scalar α and each $y \in H$ we have

$$\|y/\alpha - x\| = \|(y/\alpha + w) - z\| \geqq \|(y/\alpha + w) - w\| = \|y/\alpha\|.$$

Hence $\|y - \alpha x\| \geqq \|y\|$ and $H \perp x$.

3.2. REMARK. From 2.4 with $n = 2$ it follows that a real normed linear space E has the tri-spherical intersection property if and only if the following statement is true: If x_1, x_2 and x_3 are points in E and ρ_1, ρ_2 and ρ_3 are positive numbers with $\bigcap_{i=1}^3 S_{\rho_i}(x_i) \neq \emptyset$, then $\bigcap_{i=1}^3 S_{\rho_i}(x_i) \cap \operatorname{conv}(x_1, x_2, x_3) \neq \emptyset$. (By $\operatorname{conv}(x_1, x_2, x_3)$ we mean the smallest convex set containing each of the points x_i.)

3.3. REMARK. If the tri-spherical intersection property had been defined in 1.4 using open spheres of the form

$$S_\rho(x) = \{y \in E \mid \|x - y\| < \rho\},$$

then the statement of 3.1 remains valid. A minor modification of the proof of the implication (b)$\Rightarrow$(a) is required.

4. **Characterization of ellipsoids.** Let E be a Euclidean space. Let S be a compact convex body in E symmetric about one of its points. We call this point the centre of S. To avoid complications, we shall define an ellipsoid in E to be a convex body which determines a norm given by an inner product. We call a subset of E homothetic to S if it is of the form $\alpha S+x$ with $\alpha>0$ and $x\in E$. Our result now takes the following form: S is not an ellipsoid if and only if there are sets S_1, S_2 and S_3, with centres x_1, x_2 and x_3, homothetic to S, such that $S_1\cap S_2\cap S_3\neq\emptyset$ but $S_1\cap S_2\cap S_3\cap\operatorname{conv}(x_1, x_2, x_3)=\emptyset$.

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