ON THE INTERVAL TOPOLOGY OF AN \( l \)-GROUP

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1. Introduction. Let \( G \) be an \( l \)-group in the sense of Birkhoff [1; 2]. We consider the well-known interval topology of \( G \), which is obtained by taking the family of all closed intervals of \( G \) as a sub-base for the closed sets. Birkhoff [1, Problem 104, p. 233] raised the question whether an arbitrary partially ordered group is a topological group with respect to its interval topology. This question was answered in the negative by Northam [4], who gave an example of an \( l \)-group which is not a Hausdorff space in its interval topology (and hence not a topological group). The purpose of this note is to show that this behavior of an \( l \)-group is far from "pathological," but actually is characteristic of large classes of \( l \)-groups. Several theorems describing such classes of \( l \)-groups are obtained, all following as a consequence of a rather elementary lemma (Lemma 3 below).

2. Preliminaries. We shall employ the terminology of nets due to Kelley [3]. For brevity we shall write "\( G \) is IH" for the statement "\( G \) is a Hausdorff space in its interval topology." By a closed interval in \( G \) we shall mean any subset of the form \( \{ x \in G \mid a \leq x \leq b \} \), \( \{ x \in G \mid x \geq a \} \), or \( \{ x \in G \mid x \leq a \} \), where \( a \) and \( b \) are arbitrary elements of \( G \).

Lemma 1. If there exists a net \( \{ f(n) \mid n \in D \} \) in a partially ordered set \( G \) such that \( f(n) \) is eventually in the complement of any closed interval of \( G \), then \( G \) is not IH.

Proof. A base for the open sets of the interval topology of \( G \) consists of all subsets of the form \( \cap \{ K_i \mid i = 1, 2, \ldots, k \} \), where each \( K_i \) is the complement of a closed interval. Thus any net \( f \) satisfying the above hypothesis is eventually in each open set of \( G \). Hence \( f \) converges to each point of \( G \) and \( G \) is not IH.

Our terminology and notation for \( l \)-groups is that of Birkhoff [1; 2]. Let \( G \) be any commutative \( l \)-group and \( \mathcal{M} \) its set of meet-irreducible \( l \)-ideals. For each \( M \in \mathcal{M} \) it is known that the \( l \)-quotient-group \( G/M \) is a simply ordered group. Furthermore, \( \cap \{ M \mid M \in \mathcal{M} \} \) is empty for any commutative \( l \)-group \( G \). Hence there exists an isomorphism of \( G \) onto an \( l \)-subgroup of the direct product of a set of simply ordered groups \( \{ G_i \mid i \in I \} \), where each \( G_i \) is the \( l \)-quotient-group of \( G \) by
some meet-irreducible $l$-ideal $M$ [2, Theorem 36]. We identify any element $x$ of $G$ with the corresponding element $(x_1, x_2, \ldots, x_i, \ldots)$ of the direct product $\prod\{G_i \mid i \in I\}$. The direct product $\prod\{G_i \mid i \in I\}$ is ordered "componentwise": i.e., if $a$ and $b$ are elements of this product, we define $a \leq b$ if and only if $a_i \leq b_i$ for all $i \in I$. The identity element of $G$ will be denoted by 0, the identity of $G_i$ by $0_i$. The additive notation will be used.

The following result, which is a consequence of Theorems 23 and 27 of [2], is also known.

**Lemma 2.** If $G$ is a commutative $l$-group and $M$ is a maximal $l$-ideal of $G$, then $G/M$ is an Archimedean simply ordered group.

3. Results. Our theorems are a consequence of the following lemma.

**Lemma 3.** Let $G$ be an $l$-subgroup of the direct product $\prod\{G_i \mid i \in I\}$ of arbitrary $l$-groups. Let $r$ and $s$ be any members of the index set $I$. Suppose that there exists a net $\{f(m), m \in D\}$ of elements of $G$ satisfying

(i) for any $k \in G_r$, \( f(m) \) is eventually greater than $k$, and

(ii) $f_s(m) \leq 0$, for all $m \in D$.

Then $G$ is not IH.

**Proof.** Let $\{b(n), n \in E\}$ be any net of elements of $G$ with the property that, given any $j \in G_s$, $b_s(n)$ is eventually less than $j$. Note that for any $n \in E$, there exists an element $m_n$ in $D$ such that the $r$th component of $f(m_n) + b(n)$ is greater than 0, (one merely chooses $m_n$ so that $f_r(m_n) > -b_r(n)$). Define $g(n) = f(m_n) + b(n)$. Then $g$ is a net on $E$ to $G$ such that (i) $g_s(n)$ is eventually less than any given $j \in G_s$, and (ii) $g_r(n) > 0$, for all $n \in E$. Now consider the directed set $D \times E$ consisting of all pairs $(m, n)$ for $m \in D$, $n \in E$, directed as usual by defining $(m_1, n_1) \leq (m_2, n_2)$ if and only if $m_1 \leq m_2$ and $n_1 \leq n_2$. (We are using the same symbol $\leq$ for the order relations in $D$, $E$, and $D \times E$.) Define $h(m, n) = f(m) + g(n)$. Then $h$ is a net on $D \times E$ to $G$ which satisfies the hypothesis of Lemma 1. To see this, suppose that $J_a$ is the closed interval $\{x \in G \mid x \leq a\}$, where $a$ is an arbitrary element of $G$. Then there exists $m_0 \in D$ such that $f_r(m) > a_r$ for all $m \geq m_0$, and hence $h(m, n)$ is in the complement of $J_a$ whenever $m \geq m_0$. Likewise, given the closed interval $J'_a = \{x \in G \mid x \geq a\}$, there exists $n_0 \in E$ such that $g_s(n) < a_s$ for all $n \geq n_0$; and hence $h(m, n)$ is eventually in the complement of $J'_a$. Thus $h(m, n)$ is eventually in the complement of any closed interval, and by Lemma 1 $G$ is not IH.

Note that Lemma 3 does not require that the factor groups $G_i$ be simply ordered.
**Theorem 1.** Let $G$ be a commutative $l$-group containing a maximal $l$-ideal $M$. If there exists an element $b$ in $G$ which is incomparable with $0$ and such that $b \in M$, then $G$ is not IH.

**Proof.** By Lemma 2, $G/M$ is an Archimedean simply ordered group. Thus, since $M$ is meet-irreducible, $G$ may be considered as an $l$-subgroup of a direct product $\prod \{G_i : i \in I\}$ of simply ordered groups, where for a certain index $r \in I$, $G_r$ is Archimedean. Since $b \in M$, we have $b_r \neq 0$. Assume that $b_r > 0$, (otherwise consider $-b$). Since $b$ is incomparable with $0$, for some $s \in I$ we must have $b_s < 0$. The sequence $\{nb : n = 1, 2, \ldots\}$ then satisfies the hypothesis of Lemma 3.

**Theorem 2.** Let $G$ be an $l$-group such that $G = \prod \{G_i : i \in I\}$, where the index set $I$ contains more than one member. Then $G$ is not IH.

**Proof.** It is obviously possible to construct a net in $G$ satisfying the hypothesis of Lemma 3.

**Theorem 3.** Let $G$ be an $l$-subgroup of $\prod \{G_i : i \in I\}$, where each $G_i$ is an Archimedean simply ordered group. Then $G$ is IH if and only if $G$ is simply ordered.

**Proof.** If $b$ is an element of $G$ which is incomparable with $0$, then for some $r \in I$ we have $b_r > 0$, and for some $s \in I$ we have $b_s < 0$. The sequence $\{nb : n = 1, 2, \ldots\}$ satisfies the hypothesis of Lemma 3: hence $G$ is not IH. The converse is clear.

Since any Archimedean simply ordered group is an $l$-subgroup of the additive group of the real numbers, Theorem 3 asserts that any $l$-group of real-valued functions which is not simply ordered is not IH. This result thus includes Northam’s example (an $l$-group of continuous real-valued functions) as a special case.

It remains an open question whether Theorem 3 can be extended (at least for commutative $l$-groups) to the case where some or all of the factor groups $G_i$ are non-Archimedean. In this case there may exist no net in $G$ satisfying the conditions of Lemma 3, as the example in the next section shows.

We obtain still another application of Lemma 3. If $a$ is a positive element of the $l$-group $G$, we say that $a$ is Archimedean if and only if for any $x \in G$ there exists a positive integer $n$ with $na \geq x$. We then have

**Theorem 4.** Let $G$ be a commutative $l$-group containing an element $b$ which is incomparable with $0$ and such that $|b|$ is Archimedean. Then $G$ is not IH.
Proof. We consider $G$ as an $l$-subgroup of $\prod \{G_i | i \in I\}$, where each $G_i$ is simply ordered. For some $r \in I$, $s \in I$, we have $b_r > 0$, $b_s < 0$. Since $b_r = |b|r$, Lemma 3 may be applied to the sequence $\{nb | n = 1, 2, \ldots\}$.

4. An example. We give a simple example of a commutative $l$-group which is not simply ordered and in which there exists no net satisfying the hypothesis of Lemma 3. Let $Z$ be the integers in the usual ordering, and let $H = Z \times Z$. We order $H$ lexicographically by defining $(m_1, n_1) < (m_2, n_2)$ if and only if $m_1 < m_2$ or, when $m_1 = m_2$, if $n_1 < n_2$. Note that $H$ is non-Archimedean in this ordering. Now consider the $l$-group $H \times H$ with the usual (componentwise) direct product ordering. Define $G = \{(i, j), (m, n) \in H \times H | i = m\}$. $G$ is an $l$-subgroup of $H \times H$. Let $F = \{(i, j), (m, n) \in G | i = 0 \text{ and } m = 0\}$. Note that any element of $G$ which is not in $F$ is comparable with the identity element of $G$. Thus any net in $G$ which has one component eventually positive also has the other component eventually positive.

It should be noted, however, that $G$ is not IH. For consider the sequence defined by $f(n) = ((0, n), (0, -n))$, $n = 1, 2, \ldots$. The reader may verify that any closed interval of $G$ which contains the range of a subsequence of $f$ also contains the range of the entire sequence. We conclude that if $x$ is any member of the sequence, and $J$ is a closed interval of $G$ which does not contain $x$, then the sequence $f$ is eventually in the complement of $J$. This means that $f$ converges in the interval topology to every element $x$ in its range. Hence $G$ is not IH.

References


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