REMARKS ON A THEOREM OF M. ALTMAN

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Introduction. A mapping[1] $f: X \to X$ of a Banach space X into itself is said [1] to be locally an ϵ-mapping in the narrow sense, if for every $x \in X$ there exist two positive numbers η_x and ϵ_x such that the condition

$$x', x'' \in S(x, \epsilon_x) \quad \text{and} \quad \|f(x') - f(x'')\| < \eta_x$$

implies

$$\|x' - x''\| < \epsilon_x,$$

where $S(x, \epsilon_x)$ denotes the spherical region (in X) with centre x and radius ϵ_x.

In [1] M. Altman proved that: If the mapping $f(x) = x - F(x)$, $x \in X$, where $F(x)$ is a completely continuous transformation of the Banach space X with range in X, is locally an ϵ-mapping in the narrow sense, then the image $f(X)$ of X is an open subset of the space X.

Using this theorem he obtained in particular some conditions under which a linear mapping $f: X \to X$ of X into X is a mapping onto X.[2]

In this paper, the definition of locally an ϵ-mapping in the narrow sense is slightly modified and some theorems on mappings onto are proved for the so-called polynomial mappings.[3] It turns out that for these mappings the assumption that the mapping is locally an ϵ-mapping in the narrow sense in *every* point $x \in X$ (made in Altman's theorems referred to above) may be replaced by a much weaker assumption.

I. Definition 1. A mapping $f: X \to Y$ of a metric space X with metric ρ_1 into a metric space Y with metric ρ_2 will be called an ϵ-mapping in the narrow sense at the point $y \in f(X)$, if there *exists* a point $x \in f^{-1}(y)$ and two positive numbers η_x and ϵ_x such that

$$x', x'' \in S(x, \epsilon_x) \quad \text{and} \quad \rho_2[f(x'), f(x'')] < \eta_x$$

implies

$$\rho_1(x', x'') < \epsilon_x,$$

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[1] By "mapping" we understand always a continuous mapping.

[2] See [1, p. 1039, Corollary 1].

[3] See [5, p. 157].

where $S(x, \epsilon_x)$ denotes the spherical region (in X) with centre x and radius ϵ_x.

Note that if $f: X \to X$ is locally an ϵ-mapping in the narrow sense, then it is also an ϵ-mapping in the narrow sense at every point $y \in f(X)$—but, as can be easily seen, not conversely.

DEFINITION 2. A mapping $f: X \to Y$ of a metric space X into a metric space Y is called open at the point $y \in f(X)$ if there exists a spherical region $S(y, r)$, (in Y), with centre y and radius r such that $S(y, r) \subset f(X)$.

THEOREM 1. *If $F: X \to X$ is a completely continuous operator of a Banach space X into itself and $f(x) = x - F(x)$ is an ϵ-mapping in the narrow sense at the point $y \in f(X)$, then $f: X \to X$ is open at the point $y \in f(X)$.*

The proof of this theorem is identical with that of Theorem 1 in [1].

DEFINITION 3. A mapping $f: X \to Y$ of a metric space X into a metric space Y is called a polynomial mapping if the condition:

$\{x_n\}_{n=1,2,\dots}$ does not contain a Cauchy (fundamental) subsequence,

implies that

$\{f(x_n)\}_{n=1,2,\dots}$ also does not contain a Cauchy subsequence.

We give now some examples of polynomial mappings:

1. A mapping $f: X \to Y$ is called an ϵ-mapping in the narrow sense[4] if there exist two positive numbers η and ϵ such that the condition $\rho_2[f(x'), f(x'')] < \eta$, $x', x'' \in X$ implies $\rho_1(x', x'') < \epsilon$, where ρ_1 and ρ_2 denote the metric in X and Y respectively. We shall show that an ϵ-mapping $f: X \to Y$ in the narrow sense of a finite dimensional Banach space X into a metric space Y is a polynomial mapping.

Indeed, if $\{f(x_n)\}_{n=1,2,\dots}$ contains a Cauchy subsequence $\{f(x_n')\}_{n=1,2,\dots}$, then for any η there is $\rho_2[f(x_n'), f(x_m')] < \eta$ for n, m sufficiently large. Hence $\rho_1(x_n', x_m') = \|x_n' - x_m'\| < \epsilon$ and therefore $\{x_n'\}_{n=1,2,\dots}$ is a bounded sequence in X. Thus, X being a finite dimensional Banach space, the sequence $\{x_n'\}$ is compact (conditionally) and therefore it contains a convergent subsequence $\{x_n''\}_{n=1,2,\dots}$ which is a Cauchy subsequence of $\{x_n\}_{n=1,2,\dots}$.

2. Every mapping $f: X \to Y$ of a finite dimensional Banach space X into a finite dimensional Banach space Y, such that if $\|x_n\| \to \infty$ then also $\|f(x_n)\| \to \infty$, is a polynomial mapping.[5]

3. If there exists a constant $k > 0$ such that

[4] See [2].

[5] See [6, p. 1398, (c)].

$$\rho_2[f(x'), f(x'')] \geqq k\rho_1(x', x'')$$

then $f: X \to Y$ is a polynomial mapping (ρ_1 and ρ_2 denoting the metrics in X and Y respectively).

THEOREM 2. *If $f: X \to Y$ is a polynomial mapping of a complete metric space X into a connected metric space Y which is open at every point $y \in f(X) - J$, where $J \neq f(X)$ and $J \subset f(X)$ is a set which does not disconnect the space Y, then $f(X) = Y$.*[6]

PROOF. Since $J \neq f(X)$ and $f: X \to Y$ is open at every point $\bar{y}$ of $f(X) - J$, there exists a spherical region $S(\bar{y}, r)$ (in Y) which is contained in $f(X)$. Denote by U the union of all sets, open in Y, which are contained in $f(X)$. Evidently $\mathrm{Fr}(U) \subset \mathrm{Fr}[f(X)]$.[7] Now suppose that there exists a point $y_0 \in Y - f(X)$ and let y_1 be any point belonging to U. Since the set J does not disconnect Y, there exists in $Y - J$ a connected set C containing y_0 and y_1. But the set $\mathrm{Fr}(U)$ disconnects the space Y between y_0 and y_1;[8] therefore, there exists a point $y_2 \in [\mathrm{Fr}(U) - J] \cap C \subset \mathrm{Fr}[f(X)] - J$. By Theorem 1 of [5][9] $f(X)$ is closed in Y. Hence $y_2 \in f(X)$. By assumption, $f: X \to Y$ is open at every point $y \in f(X) - J$ in contradiction with the fact that

$$y_2 \in \mathrm{Fr}[f(X)] - J \subset f(X) - J,$$

because $f: X \to Y$ is evidently not open at every point belonging to $\mathrm{Fr}\,[f(X)]$. Thus the assumption that there exists a point $y_0 \in Y - f(X)$ leads to a contradiction. From Theorems 1 and 2 we obtain the following

THEOREM 3. *If $F: X \to Y$ is a completely continuous operator of a Banach space X into itself and $f(x) = x - F(x)$ is a polynomial mapping which is also an ϵ-mapping in the narrow sense at every point*

$$y \in f(X) - J,$$

where $J \neq f(X)$ and $J \subset f(X)$ is a set which does not disconnect X, then $f(X) = X$.

Indeed, by Theorem 1, $f: X \to X$ is open at every point $y \in f(X) - J$ and hence by Theorem 2 used for $X = Y$, there is $f(X) = X$.

REMARK 1. The function $f(x) = \arctan x$ ($F(x) = x - \arctan x$) de-

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[6] Some generalizations of this theorem to general topological spaces will be given in [6].

[7] $\mathrm{Fr}(A)$ denotes the boundary of A in Y, i.e., $\mathrm{Fr}(A) = \bar{A} \cap \overline{Y - A}$.

[8] See [3, p. 247]; also [4, p. 80].

[9] See [5, p. 158].

fined on the real line X is an example of a function which is locally an ϵ-mapping in the narrow sense, but which is not a polynomial mapping. In this case we have $f(X) \neq X$.

REMARK 2. From Theorem 3 it follows that if F satisfies the assumptions of Theorem 3, then there exists a point x such that $F(x) = x$ (i.e., $F: X \to X$ has a fixed point).

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A CHAINABLE CONTINUUM NO TWO OF WHOSE NONDEGENERATE SUBCONTINUA ARE HOMEOMORPHIC

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R. D. Anderson and Gustave Choquet [1] gave an example of a plane continuum no two of whose nondegenerate subcontinua are homeomorphic. The object of this note is to point out that there is a chainable continuum having this property. The only change we make in the construction given in [1] is to replace the n-ods used by R. D. Anderson and Gustave Choquet by chainable continua C_{n-2}.

A subcontinuum Y of a continuum X is a separating continuum of X if $X - Y$ is not connected and $\mathrm{Cl}(X - Y) = X$. A subcontinuum Y of a continuum X is a strong separating continuum if:

(1) Y is a separating continuum of X,

(2) $X - Y$ has two components, X_1 and X_2,

(3) there are points $y_1, y_2 \in Y$ such that $y_i \notin \mathrm{Cl}(X_i)$.

Let $V = \{(x, y)/y = |x| \text{ and } -1 < x < 1\}$. Let C_n be formed from n copies of V and $n+1$ "lines" so that each V is a strong separating continuum of C_n as in Figure 1 (for $n = 2$).

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