HOMOGENEOUS COUNTABLE CONNECTED HAUSDORFF SPACES

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In 1925, P. Urysohn gave an example of a countable connected Hausdorff space [4]. Other examples have been contributed by R. Bing [1], M. Brown [2], and E. Hewitt [3]. Relatively few of the properties of such spaces have been examined. In this paper the question of homogeneity is studied. Theorem I shows that there exists a bihomogeneous countable connected Hausdorff space. Theorems II and III deal with other questions related to homogeneity.

A space Z is *homogeneous* if and only if for every pair of elements x and y of Z there exists a homeomorphism f of Z onto itself such that $f(x)=y$. A space Z is *bihomogeneous* if and only if for every pair of elements x and y of Z there exists a homeomorphism f of Z onto itself such that $f(x)=y$ and $f(y)=x$.

THEOREM I. *There exists a bihomogeneous countable connected Hausdorff space.*

PROOF. Let X_0 be a countable connected Hausdorff space $\{x_1^0, x_2^0, \cdots\}$ and let $\{X_1, X_2, \cdots\}$ be a countably infinite family of mutually disjoint homeomorphic images of X_0. For each i, let h_i be a homeomorphism of X_0 onto X_i, and for each i and j, let x_j^i be $h_i(x_j^0)$.

Let p_1^1 be (x_1^2, x_2^1), and p_2^1 be (x_1^1, x_2^2). Let p_1^2 be (x_1^2, x_2^1, x_3^3), p_2^2 be (x_1^1, x_2^2, x_3^4), p_3^2 be (x_1^4, x_2^3, x_3^1), and p_4^2 be (x_1^3, x_2^4, x_3^2). Suppose that n is a positive integer such that if $i \leqq 2^n$, (1) p_i^n exists and (2) for some positive integers $k_1, k_2, \cdots$, and k_{n+1}, $p_i^n = (x_1^{k_1}, x_2^{k_2}, \cdots, x_{n+1}^{k_{n+1}})$. Then let p_i^{n+1} be $(x_1^{k_1}, x_2^{k_2}, \cdots, x_{n+1}^{k_{n+1}}, x_{n+2}^{i+2^n})$ and $p_{i+2^n}^{n+1}$ be

$$(x_1^{k_1+2^n}, x_2^{k_2+2^n}, \cdots, x_{n+1}^{2^n+k_{n+1}}, x_{n+2}^{i}).$$

For each n let Z_n be $\bigcup_{i=1}^{2^n} ([X_i - \bigcup_{j=1}^{n+1} \{x_j^i\}] \cup \{p_i^n\})$. A neighborhood system for Z_n is defined as follows: If z is a point of some X_i, then $\mathfrak{N}(z)$ denotes the neighborhood system of z in X_i. If $z \in \bigcup_{i=1}^{2^n} [X_i - \bigcup_{j=1}^{n+1} \{x_j^i\}]$, then for some i and j, $i \leqq 2^n$ and $j > n+1$, $z = x_j^i$, and in this case the neighborhood system $N^n(z)$ of z in Z_n is $\{U: U \in \mathfrak{N}(z)$ *and* $[\bigcup_{j=1}^{n+1} \{x_j^i\}] \cap U = \varnothing\}$. Suppose that $k \leqq 2^n$ and $k_1, k_2, \cdots$, and k_{n+1} are positive integers such that

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$$p_k^n = (x_1^{k_1}, x_2^{k_2}, \cdots, x_{n+1}^{k_{n+1}}).$$

If $i \leqq n+1$, let G_{k_i} be $\{U$: *$U \in \mathfrak{N}(x_i^{k_i})$ and if $r \neq i$ and $r \leqq n+1$, then $x_r^{k_i} \notin U\}$*. Then the neighborhood system $N^n(p_k^n)$ of p_k^n in Z_n is $\{[\bigcup_{i=1}^{n+1}(U_i - \{x_i^{k_i}\})] \cup \{p_k^n\} : U_i \in G_{k_i}\}$.

If n is a positive integer, $k \leqq 2^n$ and $p_k^n = (x_1^{k_1}, x_2^{k_2}, \cdots, x_{n+1}^{k_{n+1}})$, then the points $x_1^{k_1}, x_2^{k_2}, \cdots$, and $x_{n+1}^{k_{n+1}}$ are the *coordinates* of p_k^n.

LEMMA 1. *For each n, Z_n is a countable connected Hausdorff space.*

PROOF. Let n be a positive integer. It is clear that Z_n is a countable Hausdorff space. Suppose that Z_n is not connected. Then there exist disjoint nonempty open sets A and B such that $A \cup B = Z_n$. For each i, $i \leqq 2^n$, let K_i be $\{x_j^i : x_j^i \in Z_n\} \cup \{p_k^n$: *$p_k^n$ has a coordinate which belongs to $X_i\}$*. For each i, K_i is connected since it is homeomorphic with X_i. Assuming that the notation is chosen so that $K_1 \subset A$, let j be $\min\{i : K_i \subset B\}$. Now there exist integers k and r, $r < j$, such that p_k^n has a coordinate belonging to X_r and a coordinate belonging to X_j. Then $p_k^n \in K_r \cap K_j$ and hence $p_k^n \in A \cap B$. This contradicts the fact that $A \cap B = \emptyset$. This establishes the lemma.

Suppose that $n \leqq r$, U_n is an open set in Z_n, and U_r is an open set in Z_r. Then the notation

$$U_n \tilde{\subset} U_r$$

means that (1) if $p_j^n \in U_n$, then $p_j^r \in U_r$ and (2) if $x_s^t \in U_n$ then either (i) $x_s^t \in U_r$ or (ii) for some k, $p_k^r \in U_r$ and x_s^t is a coordinate of p_k^r.

Certain permutations are to be defined now. Let q_1^1 be the identity, and q_2^1 be $(1, 2)$ each on $\{1, 2\}$. Let q_1^2 be the identity, q_2^2 be $(1, 2)(3, 4)$, q_3^2 be $(1, 3)(2, 4)$, and q_4^2 be $(1, 4)(2, 3)$, each on $\{1, 2, 3, 4\}$. Suppose that n is a positive integer and if $i \leqq 2^n$, there is a permutation q_i^n on $\{1, 2, \cdots, 2^n\}$ such that for distinct positive integers $k_1, k_2, \cdots$, and k_{2^n}, $q_i^n = (k_1, k_2)(k_3, k_4) \cdots (k_{2^n-1}, k_{2^n})$. Then if $i \leqq 2^n$, $q_i^{n+1} = (k_1, k_2)(k_3, k_4) \cdots (k_{2^n-1}, k_{2^n})(k_1+2^n, k_2+2^n)(k_3+2^n, k_4+2^n) \cdots (k_{2^n-1}+2^n, k_{2^n}+2^n)$ and $q_{i+2^n}^{n+1} = (k_1, k_2+2^n)(k_3, k_4+2^n) \cdots (k_{2^n-1}, k_{2^n}+2^n)(k_1+2^n, k_2)(k_3+2^n, k_4) \cdots (k_{2^n-1}+2^n, k_{2^n})$. Notice that each permutation is the product of disjoint 2-cycles and if $r \leqq 2^n$ and $s \leqq 2^n$ then there is one and only one positive integer i such that $q_i^n(r) = s$ and $q_i^n(s) = r$.

LEMMA 2. *If $i \leqq 2^n$, $k \leqq 2^n$, and $p_k^n = (x_1^{k_1}, x_2^{k_2}, \cdots, x_{n+1}^{k_{n+1}})$, then $p_{q_i^n(k)}^n = (x_1^{q_i^n(k_1)}, x_2^{q_i^n(k_2)}, \cdots, x_{n+1}^{q_i^n(k_{n+1})})$.*

PROOF. The result is clear if $n = 1$. Assume the result holds for n and suppose that $i \leqq 2^n$, $k \leqq 2^n$, and $p_k^n = (x_1^{k_1}, x_2^{k_2}, \cdots, x_{n+1}^{k_{n+1}})$. If

$j \leqq n+1$, then $q_i^{n+1}(k_j) = q_i^n(k_j)$, $q_i^{n+1}(k_j+2^n) = q_i^{n+1}(k_j)+2^n$, $q_{i+2^n}^{n+1}(k_j) = q_i^{n+1}(k_j)+2^n$, and $q_{i+2^n}^{n+1}(k_j+2^n) = q_i^{n+1}(k_j)$. Therefore, if $m = i+2^n$, then

$$p^{n+1}_{q_i^{n+1}(k)} = (x_1^{q_i^{n+1}(k_1)}, \cdots, x_{n+1}^{q_i^{n+1}(k_{n+1})}, x_{n+2}^{q_i^{n+1}(k+2^n)}), \tag{1}$$

$$p^{n+1}_{q_i^{n+1}(k+2^n)} = (x_1^{q_i^{n+1}(k_1+2^n)}, \cdots, x_{n+1}^{q_i^{n+1}(2^n+k_{n+1})}, x_{n+2}^{q_i^{n+1}(k)}), \tag{2}$$

$$p^{n+1}_{q_m^{n+1}(k)} = (x_1^{q_m^{n+1}(k_1)}, \cdots, x_{n+1}^{q_m^{n+1}(k_{n+1})}, x_{n+2}^{q_m^{n+1}(k+2^n)}), \tag{3}$$

$$p^{n+1}_{q_m^{n+1}(k+2^n)} = (x_1^{q_m^{n+1}(k_1+2^n)}, \cdots, x_{n+1}^{q_m^{n+1}(2^n+k_{n+1})}, x_{n+2}^{q_m^{n+1}(k)}). \tag{4}$$

This establishes Lemma 2.

For each n and each i, $i \leqq 2^n$, let g_i^n be the function from Z_n onto Z_n defined as follows: If $k \leqq 2^n$ and $j > n+1$, then $g_i^n(x_j^k) = x_j^{q_i^n}(k)$ and $g_i^n(p_k^n) = p^n_{q_i^n(k)}$.

Lemma 3. *For each n and each i, $i \leqq 2^n$, g_i^n is a homeomorphism of Z_n onto Z_n.*

Proof. Suppose that $i \leqq 2^n$. It is clear that g_i^n is one-to-one and onto. Suppose that $k \leqq 2^n$, $j > n+1$ and $U \in N^n(x_j^k)$. It is clear that $g_i^n(U) \in N^n(g_i^n(x_j^k))$. Suppose that $k_1, k_2, \cdots$, and k_{n+1} are positive integers such that $p_k^n = (x_1^{k_1}, x_2^{k_2}, \cdots, x_{n+1}^{k_{n+1}})$. Let V be an element of $N^n(p_k^n)$. Then for each r, $r \leqq n+1$, there exists an element V_r of G_{k_r} such that $V = \bigcup_{r=1}^{n+1} (V_r - \{x_r^{k_r}\}) \cup \{p_k^n\}$. Now for each r, $r \leqq n+1$, $g_i^n(V_r) \in G_{q_i^n(k_r)}$. By Lemma 2,

$$g_i^n(p_k^n) = (x_1^{q_i^n(k_1)}, \cdots, x_{n+1}^{q_i^n(k_{n+1})}),$$

and hence

$$g_i^n(V) = \bigcup_{r=1}^{n+1} (g_i^n(V_r) - \{x_r^{q_i^n(k_r)}\}) \cup \{p^n_{q_i^n(k)}\}$$

which is an element of $N^n(g_i^n(p_k^n))$. This shows that $(g_i^n)^{-1}$ is continuous. Since $g_i^n = (g_i^n)^{-1}$ it follows that g_i^n is a homeomorphism.

Lemma 4. *For each n, r, and s such that $r \leqq 2^n$ and $s \leqq 2^n$, there exists one and only one positive integer i, $i \leqq 2^n$, such that $g_i^n(p_s^n) = p_r^n$ and $g_i^n(p_r^n) = p_s^n$.*

Proof. Suppose that $r \leqq 2^n$ and $s \leqq 2^n$. Now there exists one and only one positive integer i, $i \leqq 2^n$, such that $q_i^n(r) = s$ and $q_i^n(s) = r$. Then $g_i^n(p_r^n) = p^n_{q_i^n(r)} = p_s^n$ and $g_i^n(p_s^n) = p^n_{q_i^n(s)} = p_r^n$.

For each k, let p_k be the sequence whose ith term is, for sufficiently large n, the ith coordinate of p_k^n. Let Z be $\{p_k$: *k is a positive integer*$\}$. A neighborhood system for Z is defined as follows: If for some i, U_i is an open set in Z_i let U_i^* denote the set of all points p_j of Z such that either (1) $p_j^i \in U_i$ or (2) p_j has a coordinate in U_i. Now for each k, $N(p_k)$, the neighborhood system of p_k in Z is $\{(\bigcup_{i=n}^{\infty} U_i^*): k \leqq 2^n$, *$U_i$ is an open set in Z_i, $p_k^i \in U_i$, and $U_i \tilde{\subset} U_{i+1}$*$\}$.

For each r, let g_r be the function from Z onto Z defined as follows: Suppose that $k \leqq 2^n$. For some s, $s \leqq 2^n$, $g_r^n(p_k^n) = p_s^n$. Set $g_r(p_k) = p_s$. Then g_r is well defined since if $t \geqq n$, $g_r^t(p_k^t) = p_s^t$.

It follows from Lemma 4 and the definition of g_i that for each r and each s there exists one and only one positive integer i such that $g_i(p_r) = p_s$ and $g_i(p_s) = p_r$.

LEMMA 5. *Z is a countable connected Hausdorff space.*

PROOF. Z is clearly a countable Hausdorff space. Suppose that Z is not connected. Then there exist disjoint nonempty open sets A and B such that $A \cup B = Z$. There exist positive integers k and j such that $p_k \in A$ and $p_j \in B$. Let n be a positive integer such that $k \leqq 2^n$ and $j \leqq 2^n$. Let A_n be $\{p_i^n: p_i \in A\} \cup \{x_i^s$: *$s \leqq 2^n$ and there exists an element p_i of A such that x_i^s is a coordinate of p_i*$\}$. Let B_n be $\{p_i^n: p_i \in B\} \cup \{x_i^s$: *$s \leqq 2^n$ and there exists an element p_i of B such that x_i^s is a coordinate of p_i*$\}$. Now A_n and B_n are nonempty disjoint open sets in Z_n, and $A_n \cup B_n = Z_n$. This contradicts the fact that Z_n is connected, and hence Z is connected.

LEMMA 6. *For each r, g_r is a homeomorphism of Z onto Z.*

PROOF. Let r be a positive integer. It is clear that g_r is one-to-one and onto. Let k be a positive integer. There exists a j such that $g_r(p_k) = p_j$. Suppose that $U \in N(p_k)$. Now there exists an n and sets $U_n, U_{n+1}, \cdots$ such that if $s \geqq n$, U_s is open in Z_s, $p_k^s \in U_s$, $U_s \tilde{\subset} U_{s+1}$, and such that $U = \bigcup_{s=n}^{\infty} U_s^*$. Let m be a positive integer such that $j \leqq 2m$. Now since $U_s \tilde{\subset} U_{s+1}$, then $\bigcup_{s=n}^{\infty} U_s^* = \bigcup_{s=m}^{\infty} U_s^*$, and hence $U = \bigcup_{s=m}^{\infty} U_s^*$. Now for each s, $s \geqq m$, g_r^s is a homeomorphism and hence $g_r^s(U_s) \in N^s(p_j^s)$. Also $g_r^s(U_s) \tilde{\subset} g_r^{s+1}(U_{s+1})$. Now since $g_r(U_s^*) = [g_r^s(U_s)]^*$ and $g_r(U) = \bigcup_{s=m}^{\infty} g_r(U_s^*)$, then $g_r(U) \in N(p_j)$. This shows that $(g_r)^{-1}$ is continuous. Since $g_r = (g_r)^{-1}$, g_r is continuous and it follows that g_r is a homeomorphism.

The above lemmas establish the theorem.

THEOREM II. *There exists a countable connected Hausdorff space Z such that each point of Z is a cut point of Z.*

PROOF. Let Y be a countable connected Hausdorff space and let $\{X_{ij}$: *each of i and j is a positive integer*$\}$ be a countably infinite family of mutually disjoint countable connected Hausdorff spaces, each disjoint from Y. Let y^1 be a one-to-one sequence onto Y, i.e., let a fixed denumeration $y_1^1, y_2^1, y_3^1, \cdots$ be placed on the elements of Y, and for each (i, j) let x^{ij} be a one-to-one sequence onto X_{ij}.

If $p \in Y$, then $\mathfrak{N}(p)$ denotes the neighborhood system of p in Y, and if p is a point of some X_{ij}, then $\mathfrak{N}(p)$ denotes the neighborhood system of p in X_{ij}.

Let Z_1 be $\bigcup_{i=1}^{\infty} (X_{1i} - \{x_1^{1i}\}) \cup \{(y_i^1, x_1^{1i})$: *i is a positive integer*$\}$. A neighborhood system for Z_1 is defined as follows: If $z \in \bigcup_{i=1}^{\infty} (X_{1i} - \{x_1^{1i}\})$, then for some i and some k, $k \neq 1$, $z = x_k^{1i}$, and in this case $N^1(x_k^{1i})$, the neighborhood system of x_k^{1i} in Z_1, is $\{U$: $U \in \mathfrak{N}(x_k^{1i})$ *and* $x_1^{1i} \notin U\}$. In order to define neighborhoods in Z_1 of the remaining points of Z_1, certain functions will be introduced. Suppose that i is a positive integer and that $V \in \mathfrak{N}(y_i^1)$. Let V^* be $\{(y_i^1, x_1^{1i}): y_i^1 \in V\}$. V^* is a one-to-one function. Let $W(V)$ be the set of all functions w on (range V^*) such that if $x_1^{1i} \in$(range V^*), then $w(x_1^{1i}) \in \mathfrak{N}(x_1^{1i})$. If $w \in W(V)$ let $U(V, w)$ be

$$V^* \cup [\bigcup \{[w(x_1^{1i}) - \{x_1^{1i}\}]: x_1^{1i} \in \operatorname{dom} w\}].$$

Now if for some i, $z = (y_i^1, x_1^{1i})$, then $N^1(z)$, the neighborhood system in Z_1 of z, is $\{U(V, w)$: $V \in \mathfrak{N}(y_i^1)$ *and* $w \in W(V)\}$. Z_1 with the resulting topology is a countable connected Hausdorff space.

LEMMA 1. *For each i, (y_i^1, x_1^{1i}) is a cut point of Z_1.*

PROOF. Let i be a positive integer. Let A be $\{x_j^{1i}: j > 1\}$. Then A and its complement relative to $Z_1 - \{(y_i^1, x_1^{1i})\}$ are disjoint nonempty open sets whose union is $Z_1 - \{(y_i^1, x_1^{1i})\}$. This establishes Lemma 1.

Now for each n, $n \neq 1$, let y^n be a one-to-one sequence onto $\bigcup_{i=1}^{\infty} (X_{(n-1)i} - \{x_1^{(n-1)i}\})$, and let Z_n be $\bigcup_{i=1}^{\infty} (X_{ni} - \{x_1^{ni}\}) \cup \{(y_k^j, x_1^{jk})$: *each of j and k is a positive integer and* $j \leq n\}$. A neighborhood system for Z_n is defined as follows: If $z \in \bigcup_{i=1}^{\infty} (X_{ni} - \{x_1^{ni}\})$, then for some i and some k, $k \neq 1$, $z = x_k^{ni}$, and in this case $N^n(x_k^{ni})$, the neighborhood system in Z_n for x_k^{ni}, is $\{U$: $U \in \mathfrak{N}(x_k^{ni})$ *and* $x_1^{ni} \notin U\}$. In order to define neighborhoods in Z_n of the remaining points of Z_n, certain functions will be introduced. Suppose that i is a positive integer and that $V \in N^{n-1}(y_i^n)$. Let V^* be $\{(y_j^n, x_1^{nj}): y_j^n \in V\}$. V^* is a one-to-one function. Let $W(V)$ be the set of all functions w on (range V^*) such that if $x_1^{nj} \in$ (range V^*), then $w(x_1^{nj}) \in \mathfrak{N}(x_1^{nj})$. If $w \in W(V)$, let $U(V, w)$ be $V^* \cup [\bigcup\{[w(x_1^{nj}) - \{x_1^{nj}\}]: x_1^{nj} \in \operatorname{dom} w\}]$. Now if for some i, z

$=(y_i^n, x_1^{ni})$, then $N^n(z)$, the neighborhood system in Z_n of z is $\{U(V, w): V\in N^{n-1}(y_i^n) \text{ and } w\in W(V)\}$. Suppose that i and j are positive integers, $j\leqq n-1$, and that $V\in N^{n-1}(y_i^j, x_1^{ji})$. Let V^* be $\{(y_s^r, x_1^{rs}): (y_s^r, x_1^{rs})\in V \text{ or } y_s^r\in V\}$. Let $W(V)$ be the set of all functions w on $\{x_1^{ns}: (y_s^n, x_1^{ns})\in V^*\}$ such that if $x_1^{ns}\in\{x_1^{ns}: (y_s^n, x_1^{ns})\in V^*\}$, then $w(x_1^{ns})\in\mathfrak{N}(x_1^{ns})$. If $w\in W(V)$, let $U(V, w)$ be

$$V^*\cup[\cup\{[w(x_1^{ns})-\{x_1^{ns}\}]: x_1^{ns}\in \operatorname{dom} w\}].$$

Now if for some i and some j, $j\leqq n-1$, $z=(y_i^j, x_1^{ji})$, then $N^n(z)$, the neighborhood system in Z_n of z, is $\{U(V, w): V\in N^{n-1}(y_i^j, x_1^{ji}) \text{ and } w\in W(V)\}$. Now for each n, Z_n, with the resulting topology, is a countable connected Hausdorff space.

LEMMA 2. *For each i, j, and n, $i\leqq n$, (y_j^i, x_1^{ij}) is a cut point of Z_n.*

PROOF. Lemma 1 shows that the conclusion is true if $n=1$. Let n be a positive integer greater than 1, and suppose that if k is a positive integer less than n and i and j are integers, $i\leqq k$, then (y_j^i, x_1^{ij}) is a cut point of Z_{n-1}. Let i and j be positive integers, $i\leqq n$.

Now if $i<n$, there exist disjoint nonempty open subsets A and B of $Z_{n-1}-\{(y_j^i, x_1^{ij})\}$ such that $A\cup B=Z_{n-1}-\{(y_j^i, x_1^{ij})\}$. Let J_0 be a subset of the positive integers such that $k\in J_0$ if and only if $y_k^n\in A$. Let C be $\{(y_s^r, x_1^{rs}): (y_s^r, x_1^{rs})\in A \text{ or } y_s^r\in A\}\cup[\cup\{(X_{nk}-\{x_1^{nk}\}): k\in J_0\}]$, and D be $\{(y_s^r, x_1^{rs}): (y_s^r, x_1^{rs})\in B \text{ or } y_s^r\in B\}\cup[\cup\{(X_{nk}-\{x_1^{nk}\}): k\notin J_0\}]$. Now C and D are disjoint nonempty open subsets of $Z_n-\{(y_j^i, x_1^{ij})\}$ such that $C\cup D=Z_n-\{(y_j^i, x_1^{ij})\}$.

If $i=n$, the proof that (y_j^i, x_1^{ij}) is a cut point of Z_n is similar to the proof that (y_j^1, x_1^{1j}) is a cut point of Z_1, and hence is omitted.

Let Z be $\{(y_j^i, x_1^{ij}): \text{each of } i \text{ and } j \text{ is a positive integer}\}$. A neighborhood system for Z is defined as follows: Suppose that $z\in Z$. Then for some i and some j, $z=(y_j^i, x_1^{ij})$. Let $S(z)$ be the set of all functions V on $\{i, i+1, i+2, \cdots\}$ such that (1) $V_i\in N^i(z)$ and (2) if $k>i$, then $V_k\in N^k(z)$ and for some element w of $W(V_{k-1})$, $V_k=U(V_{k-1}, w)$. Now if for some i and some j, $z=(y_j^i, x_1^{ij})$, then $N(z)$, the neighborhood system in Z of z is $\{[\cup_{k=i}^{\infty}(V_k\cap Z)]: V\in S(z)\}$. Z with the resulting topology is a countable connected Hausdorff space.

LEMMA 3. *Each point of Z is a cut point of Z.*

PROOF. Suppose that $z\in Z$. Then there exist integers i and j such that $z=(y_j^i, x_1^{ij})$. Now there exist sequences $A_i, A_{i+1}, A_{i+2}, \cdots$ and $B_i, B_{i+1}, B_{i+2}, \cdots$ such that for every positive integer k greater than $i-1$, (1) A_k and B_k are nonempty disjoint open subsets of Z_k

$-\{(y_j^t, x_1^{tj})\}$ such that $A_k \cup B_k = Z_k - \{(y_j^t, x_1^{tj})\}$, (2) if $v \in A_k$ and V is an element of $N^k(v)$ such that $V \subset A_k$, then for every element w of $W(V)$, $U(V, w) \subset A_{k+1}$, and (3) if $v \in B_k$ and V is an element of $N^k(v)$ such that $V \subset B_k$, then for every element w of $W(V)$, $U(V, w) \subset B_{k+1}$. Let A be $\bigcup_{k=t}^{\infty} (A_k \cap Z)$ and B be $\bigcup_{k=t}^{\infty} (B_k \cap Z)$. Then A and B are nonempty disjoint open subsets of $Z - \{(y_j^t, x_1^{tj})\}$ such that $A \cup B = Z - \{(y_j^t, x_1^{tj})\}$.

The above lemmas establish the theorem.

THEOREM III. *There exists a countable connected Hausdorff space Z such that the only homeomorphism of Z onto itself is the identity.*

OUTLINE OF PROOF. If Z is a connected space and $z \in Z$, then the statement that z has component number n means that n is a positive integer and that $Z - \{z\}$ has exactly n components. Using a construction similar to that of Theorem II, a countable connected Hausdorff space Z can be constructed such that (1) every point of Z is a cut point of Z and (2) distinct points of Z have distinct component numbers. It follows that the only homeomorphism of Z onto itself is the identity.

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