A MAXIMAL THEOREM

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Let $X$ denote the unit circle and $L^p$, $1 < p < \infty$, the usual Lebesgue space. Given $f \in L^p$ there is a harmonic function $u$ in the unit disc with $L^p$ boundary value $f$. Set $f^*(x) = \sup_{r < 1} |u(r, x)|$. The Hardy-Littlewood Maximal Theorem asserts that there exists a constant $B_p$ such that $\|f^*\|_p \leq B_p \|f\|_p$. A similar theorem is given in higher dimensions by H. E. Rauch [2] and K. T. Smith [3] where $X$ is now the unit sphere in $n$-space. These results are obtained by first proving a maximal ergodic theorem and then passing over to the maximal theorem. The purpose of this note is to remark that the maximal theorem is a trivial deduction from a maximal ergodic theorem which is itself completely standard, so that, in effect, there is very little to prove.

Before presenting the general procedure, I give an example which illustrates everything. Let $X$ be the real line and take $f \in L^p$. The harmonic function in the upper half plane with boundary values $f$ is

$$h(t, x) = \int_{-\infty}^{\infty} Q(t, x - y)f(y)dy$$

where $Q$ is the Poisson kernel, $Q(t, x) = \pi^{-1}t(t^2 + x^2)^{-1}$, $t > 0$. We set $f^*(x) = \sup_{t > 0} |h(t, x)|$ and the relevant maximal theorem is $\|f^*\|_p \leq B_p \|f\|_p$. The only fact we need about the Poisson kernel is that the convolution operators $Q(t)$ form a semi-group having the symbolic form $Q(t) = \exp(-t\Lambda^{1/2})$ where $\Lambda^{1/2}$ is the positive square root of $\Lambda = -d^2/dx^2$. Now $P(t) = \exp(-t\Lambda)$ is a formal expression for the Gaussian semi-group of convolution operators having the Weierstrass kernel, $P(t, x) = (1/2)\pi^{-1/2}t^{-1/2} \exp\{-t^{-1}x^2\}$. Put

$$g(t, x) = \int_{-\infty}^{\infty} P(t, x - y)f(y)dy.$$ We evidently have the relation

$$h(s, x) = \int_{-\infty}^{\infty} \phi(s, t)g(t, x)dt$$

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1 Research for this paper was supported by the NSF Contract No. G5253.
2 The original Hardy-Littlewood paper is in Acta Math. vol. 54 (1930) pp. 81–116. A convenient reference is [4, pp. 244–247]. Both this theorem and the generalization to spheres are listed as exercises in [1, Exercises 7 and 8, p. 718].
where $\phi$ is determined by the equation
\[
\exp(-s\lambda^{1/2}) = \int_0^\infty \phi(s, t) \exp(-t\lambda)\,dt.
\]

It is to be noted that $\phi$ is independent of the explicit nature of the kernels $P$ and $Q$. One easily calculates that $t\phi(s, t) = \psi(s^{-2}t)$ where
\[
\psi(u) = \pi^{-1/2}(4u)^{-1/2} \exp\{-u^{-1}\}.
\]
What is important is that we have

(i) $\int_0^t |\phi(s, t)|\,dt \leq I,$

(ii) the total variation in $t$ of $t\phi(s, t)$ is not greater than $V,$

where $I$ and $V$ are constants independent of $s$. Now one has
\[
|\tilde{h}(s, x)| \leq \int_0^\infty |\phi(s, t)| \left| g(t, x) \right|\,dt.
\]

Assume that $\phi$, or some majorant of $|\phi|$, satisfies (i) and (ii), and set
\[
a(t, x) = t^{-1} \int_0^\infty |g(u, x)|\,du,
\]
and put $\tilde{f}(x) = \sup_{t>0} a(t, x)$. It is evident that
\[
|\tilde{h}(s, x)| \leq \int_0^\infty |\phi(s, t)| \, dt a(t, x) = \int_0^\infty a(t, x) |\phi(s, t)|\,dt
\]
\[
+ \int_0^\infty |t\phi(s, t)| \, dt a(t, x) \leq (I + 2V)\tilde{f}(x).
\]

Thus one concludes that $\|\tilde{f}^*\|_p \leq (I + 2V)\|\tilde{f}\|_p$. Next, we observe that $\tilde{f}$ is simply the supremum of the averages of $|f|$ with respect to a probability semi-group corresponding to a measure-preserving flow on $X$, in this case the flow is Brownian motion. Hence the maximal ergodic theorem\(^8\) is applicable; it says that for $1 < p < \infty$, $\|f\|_p \leq A_p\|f\|_p$. The maximal theorem, $\|f^*\|_p \leq B_p\|f\|_p$, follows with $B_p = (I + 2V)A_p$.

The specific deduction made thus far has little merit; the standard derivation of the Hardy-Littlewood Maximal Theorem, of [4, p. 245] uses the same reasoning except that the underlying flow is uniform.

\(^8\) Cf. [1, Chapter VIII, especially Theorem 7, p. 693]. Our indebtedness to the ideas there (which appeared earlier in J. Rat. Mech. Anal. vol. 5 (1956) pp. 129–178) is evident.
A maximal theorem rather than Brownian motion. However, it is clear that the argument persists in a quite general context, and we intend to put this generality to use.

Take for \( X \) any set, \( \mathcal{F} \) a Borel field of subsets of \( X \), and \( \mu \) a completely additive measure defined on \( \mathcal{F} \). By a measure-preserving flow on \( X \) we mean the assignment to each \( t > 0 \) and \( E \in \mathcal{F} \) of a subset \( E_t \) of \( X \), measurable with respect to the canonical extension of \( \mu \), such that the measure of the symmetric difference of \( E_{t+h} \) and \( E_t \) tends to zero as \( h \) tends to zero from above. Let \( L^p \) be the Lebesgue space with respect to the measure \( \mu \); we confine our attention to the range \( 1 < p < \infty \). The flow induces a strongly continuous semi-group of operators \( \{ P(t) \} \) or \( L^p \). Given \( f \in L^p \) we form \( g(t, x) = P(t)f(x) \) and \( \bar{f}(x) = \sup_{t \geq 0} t^{-1} \int_0^t \| g(u, x) \| du \). The maximal ergodic theorem states that there exist constants \( A_p \) such that \( \| \bar{f} \| \leq A_p \| f \| \). The operators \( P(t) \) can be considered to be defined for all \( L^p \) spaces simultaneously; for brevity we shall call \( \{ P(t) \} \) a probability semi-group. What we have proved above is

**Theorem.** Let \( \{ P(t) \} \) be a probability semi-group defined on a measure space \( X \) and suppose \( \{ Q(s) \} \), \( s > 0 \), is a one-parameter family of operators subordinate to \( \{ P(t) \} \), i.e., \( Q(s) = \int_0^s \phi(s, t)P(t)dt \). Let \( L^p \) be the Lebesgue space with respect to an invariant measure, and for \( f \in L^p \) set \( h(s, x) = Q(s)f(x) \), \( f^*(x) = \sup_{s > 0} | h(s, x) | \). If the subordinator \( \phi(s, t) \), or some majorant of \( | \phi | \), satisfies (i) and (ii) above then for \( 1 < p < \infty \) there exists a constant \( B_p \) such that \( \| f^* \| \leq B_p \| f \| \).

For an application consider the unit sphere \( X \) in \( n \)-dimensional space and the \( L^p \) spaces with respect to the uniform measure. Given \( f \in L^p \) we let \( u(r, x) \) be the function harmonic in \( r < 1 \) with boundary values \( f(x) \). Thus \( \Delta u = 0 \) where \( \Delta \) is the Laplacian. We may write \( \Delta = -r^{-n}(\partial/\partial r)(r^n(\partial/\partial r)) + r^{-2}A \) where \( A \) is the Beltrami operator on the unit sphere. It follows that \( r \partial u/\partial r = \{ (\Delta + c^2)^{1/2} - c \} u \) where \( n = 2c + 1 \). Set \( h(s, x) = u(\exp(-s), x) \); then \( h(s, x) = Q(s)f(x) \) where \( Q(s) \) has the symbolic form \( Q(s) = \exp \{ -s[\lambda + c^2]^{1/2} - c \} \). Let \( \{ P(t) \} \) be the semi-group \( P(t) = \exp(-t\Lambda) \). This is a probability semi-group corresponding to Brownian motion on the sphere which is a measure-preserving flow with respect to the uniform measure. \( Q(s) = \int_0^s \phi(s, t)P(t)dt \) where the subordinator \( \phi \) is determined by \( \exp \{ -s[\lambda + c^2]^{1/2} - c \} = \int_0^s \phi(s, t)exp(-t\lambda)dt \). Using the calculation given above we find \( t\phi(s, t) = \exp(cs - c^2t)\psi(s^{-2}t) \) whence it follows that (i) and (ii) are satisfied. The result is

**Corollary.** Suppose \( f \in L^p \) on the unit sphere and \( u(r, x) \) is the
function harmonic in the unit ball with boundary values $f(x)$. Then if $1 < p < \infty$ there exists a constant $B_p$ such that $\|f^*\|_p \leq B_p\|f\|_p$ where $f^*(x) = \sup_{r<1} |u(r,x)|$.

We turn now to the question of the validity of the maximal theorem for rather general probability semi-groups. Suppose \{P(t)\} is a probability semi-group with symbolic form $P(t) = \exp(-t\Lambda)$. If $\chi$ is a function alternating of order $\infty$, i.e., a completely monotone mapping, on $(0, \infty)$ and $\chi(0) = 0$ then \{Q(t)\} is again a probability semi-group where $Q(s) = \exp\{-s\chi(\Lambda)\}$. A rigorous discussion of the infinitesimal generators $\Lambda$ and $\chi(\Lambda)$ is irrelevant here since what is meant is that $Q(s) = \int_0^s \phi(s, t)P(t)dt$ where $\phi$ is determined by $\exp\{-s\chi(\Lambda)\} = \int_0^s \phi(s, t)\exp(-t\Lambda)dt$. (In general we should write $d_t\Phi(s, t)$ for $\phi(s, t)dt$, where for each $s$, $\Phi(s, t)$ is a function increasing from 0 to 1 on $[0, \infty]$.) Such a process of subordination takes probability semi-groups into probability semi-groups. We can assert the maximal theorem for the semi-group \{Q(t)\} if $\phi$ satisfies (i) and (ii) above, (i) is trivial since $\phi(s, t) = 0$ and $\int_0^s \phi(s, t)dt = 1$. It remains to decide for what $\chi$'s (ii) holds. We shall now show that for $\chi(x) = x^c$, $0 < c < 1$, (ii) is valid. Here $\int_0^s \phi(s, t)\exp(-t\Lambda)dt = \exp(-s\lambda^c)$ so that $t\phi(s, t) = \psi_e(s^{-1/c}t)$ where $\psi_e$ is determined by

$$\int_0^\infty \psi_e(u)\exp(-u\lambda)du = c\lambda^{e-1}\exp(-\lambda^e).$$

(The $\psi$ used above corresponds to $\varepsilon = 1/2$.) Inversion of Laplace transforms gives

$$\psi_e(u) = \pi^{-1}\int_0^\infty \exp\{x^{1/c}u \cos \theta - x \cos \theta c\} \cdot \sin\{x^{1/c}u \sin \theta - x \sin \theta c + \theta c\}dx$$

where $\theta$ may be chosen at will in the range $\pi/2 \leq \theta \leq \pi$. Changing variables we calculate the derivative as

$$\psi'_e(u) = c\pi^{-1}\int_0^\infty \exp\{yu \cos \theta - y^e \cos \theta c\} \cdot \sin\{yu \sin \theta - y^e \sin \theta c + \theta(c + 1)\}y^c dy.$$ 

Taking $\theta = \pi$ we have the estimate for $u > 2$:

$$|\psi'_e(u)| \leq c\pi^{-1}\int_0^\infty \exp\{-\frac{1}{2} yu\}y^c dy = O(u^{-1-c}).$$

Taking $\theta = \pi/2$, we obtain the uniform bound
\[ |\psi'(u)| \leq c\pi^{-1} \int_0^\infty \exp\{-y^c \cos(1/2)\pi c\} y^c dy. \]

Therefore \( \int_0^\infty |\psi'(u)| du = V < \infty \). The total variation of \( t\phi(s, t) \) is \( V \), and so (ii) holds.

The reasoning of the last paragraph establishes

**Theorem.** Suppose there is a measure-preserving flow on a measure space \( X \) inducing probability semi-groups \( \{Q(t)\}, t > 0 \), on the Lebesgue spaces \( L^p \) with respect to an invariant measure. If the family of operators \( \{Q(t)\} \) is subordinate to a probability semi-group \( \{P(t)\} \) via the formal relation \( Q(t) = \exp(-t\Delta^c) \), \( P(t) = \exp(-t\Delta) \), where \( 0 < c < 1 \), then given \( f \in L^p \) and \( f^*(x) = \sup_{t > 0} |Q(t)f(x)| \) we have for \( 1 < p < \infty \), \( \|f^*\|_p \leq B_p \|f\|_p \) where the bounds, \( B_p \), depend only on \( p \) and \( c \).

**Bibliography**