ON THE EQUATION $ax-xb=c$ IN DIVISION RINGS

G. H. MEISTERS[1]

Consider the two term linear equation

$$ax - xb = c \tag{1}$$

for a, b and c in a ring R. If R is a field then (1) has a solution x in R for each c in R provided $a \neq b$. In this paper we ask if there are rings other than fields for which the conclusion of the last statement is true. Thus we consider the condition

(C) *$a, b, c \in R$ and $a \neq b$ implies $ax-xb=c$ has a solution x in R,*

for rings R.

LEMMA 1. *If a ring R satisfies* (C) *then R is a division ring.*

PROOF. If R satisfies (C) then, for each nonzero element a in R the equations $ax=c$ and $xa=c$ have solutions x in R, and so R is a division ring as is shown in many places. (For example, see page 87 of [7].)

In view of this lemma our question reduces to

QUESTION 1. *Can a noncommutative division ring satisfy condition* (C)?

The following lemma shows that if condition (C) is slightly modified, then the answer to the corresponding question is "no."

LEMMA 2. *A ring R is a field if and only if* (C*): *$a, b, c \in R$ and $a \neq b$ implies $ax-xb=c$ has a unique solution x in R.*

PROOF. If R is a field then (C*) is clearly true. On the other hand suppose (C*) is true. Then by Lemma 1 R is a division ring. If R is not a field there must exist two distinct nonzero elements α and β in R such that $\alpha\beta \neq \beta\alpha$. But then the equation

$$(\alpha\beta)x - x(\beta\alpha) = 0$$

must have a unique solution x in R. However, $x=0$ and $x=\beta^{-1}$ are clearly two distinct solutions, which contradicts (C*). Consequently, R must be a field.

The following two theorems give a negative answer to Question 1

Presented to the Society, November 19, 1960; received by the editors March 3, 1960 and, in revised form, May 31, 1960.

[1] The author wishes to thank Dr. J. M. Anderson and Professor Trevor Evans for many encouraging and helpful discussions.

for division rings which are finite dimensional, or algebraic and separable, over their center.

THEOREM 1. *If a division ring Δ is finite dimensional over its center F and if Δ satisfies* (C), *then Δ is commutative.*

PROOF. If Δ is not commutative then there exist two distinct nonzero elements, say α and β, such that $\alpha\beta\neq\beta\alpha$. Now Δ is a finite dimensional vector space over its center F, and the function $L(x)=\alpha\beta x-x\beta\alpha$ is a linear map of this vector space into itself. $L(x)$ is not one-to-one since $L(0)=L(\beta^{-1})=0$. But a linear transformation of a finite dimensional vector space into itself is one-to-one if and only if it is *onto*. Hence $L(x)$ does not map Δ onto itself. This means that the equation

$$L(x) \equiv \alpha\beta x - x\beta\alpha = c$$

does not have a solution x in Δ for each c in Δ, contrary to the hypothesis that Δ satisfies (C). Hence Δ is commutative.

THEOREM 2. *If Δ is a division ring algebraic and separable over its center F and if Δ satisfies* (C) *then Δ is commutative.*

PROOF. Suppose Δ is not commutative. Then there exist two distinct nonzero elements, say α and β, such that $\alpha\beta\neq\beta\alpha$. Since Δ satisfies (C) the equation

$$\alpha\beta x - x\beta\alpha = c \tag{2}$$

must have a solution x in Δ for each c in Δ. Let ξ denote a solution of (2) for $c=\alpha$. Then $x=\xi\alpha^{-1}$ is a solution of the equation

$$\alpha\beta x - x\alpha\beta = 1. \tag{3}$$

Consequently, $\alpha\beta\notin F$. However, since Δ is algebraic over F, $\alpha\beta$ must be a root of some polynomial $p(\lambda)\in F[\lambda]$. Let $m(\lambda)$ denote the minimum polynomial of $\alpha\beta$ in $F[\lambda]$. That is, $m(\lambda)$ is to denote the unique reduced polynomial in $F[\lambda]$ of minimum degree for which $m(\alpha\beta)=0$. (Our terminology here is that of [5].) Then $m(\lambda)\equiv a(\lambda)(\lambda-\alpha\beta)$, where

$$a(\lambda) = \alpha_n\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$$

and $\alpha_n, \cdots, \alpha_0$ are in Δ. Now applying Theorem 2 of [5], equation (3) has a solution x in Δ if and only if $\alpha\beta$ is a right root of $a(\lambda)$. But since (3) has the solution $x=\xi\alpha^{-1}$ it follows that $\alpha\beta$ is a right root of $a(\lambda)$, where

$$m(\lambda) \equiv a(\lambda)(\lambda - \alpha\beta),$$

and this contradicts the hypothesis that Δ is separable over F. Hence Δ is commutative.

If there exists a noncommutative division ring Δ which satisfies condition (C) it must, because of Theorem 1, be infinite dimensional over its center. If one seeks a noncommutative Hilbert division ring $\Gamma\langle t; s\rangle$ of formal power series $\sum_{i>n}^{\infty} \theta_i t^i$ (see [4, p. 187]) which satisfies (C), and if one does not already possess an example of a noncommutative division ring Γ which satisfies (C), then one must construct $\Gamma\langle t; s\rangle$ from a *field* $\Gamma = F$. This is so because if $\Gamma\langle t; s\rangle$ satisfies (C) then it follows easily that the division sub-ring Γ must satisfy (C). The following theorem (Theorem 3) will allow us to prove (see corollary of Theorem 4) that no noncommutative Hilbert division ring $F\langle t; s\rangle$ constructed from a field F can satisfy condition (C), thus giving a negative answer to Question 1 for this class of infinite dimensional division rings.

THEOREM 3. *If Δ is a division ring satisfying* (C), *then every nonzero inner-derivation of Δ maps Δ onto itself.*

PROOF. If Δ is commutative, then the conclusion of the theorem is vacuously satisfied. If Δ is not commutative there exists an element d in Δ which is not in the center F of Δ, and $(d): x \to dx - xd$ is a nonzero inner-derivation of Δ. Furthermore, every nonzero inner-derivation of Δ corresponds to such an element d. Now $d \notin F$ implies the existence of an element $\alpha \notin F$ such that $\alpha d \neq d\alpha$. Let $\beta = \alpha^{-1} d$. Then $\alpha\beta \neq \beta\alpha$ and

$$dx - xd \equiv \alpha\beta x - x\alpha\beta.$$

The equation

$$dx - xd \equiv \alpha\beta x - x\alpha\beta = c \tag{4}$$

is equivalent to the equation

$$\alpha\beta y - y\beta\alpha = c\beta^{-1}, \tag{5}$$

where $y = x\beta^{-1}$. However, since $\alpha\beta \neq \beta\alpha$, it follows from (C) that (5) has a solution y for every c in Δ, and so (4) has the solution $x = y\beta$ for every c in Δ. This completes the proof.

In [3] Bruno Harris gave an example of a division ring D every element c of which is a commutator $[x, y]$ for some x and y in D. A division ring Δ which possesses even one inner-derivation mapping Δ onto itself would certainly be another such example. This raises the following questions.

QUESTION 2. *Does there exist a division ring Δ which possesses an inner-derivation mapping Δ onto itself?*

QUESTION 3. *Does there exist a noncommutative division ring Δ such that every nonzero inner-derivation of Δ maps Δ onto itself?*

The following theorem gives a negative answer to Question 3 for the class of Hilbert division rings $F\langle t;\, s\rangle$ which are constructed from a field F.

THEOREM 4. *Let F be a field and let s be an automorphism of F other than the identity automorphism. Then the Hilbert division ring $F\langle t;\, s\rangle$ always possesses a nonzero inner-derivation which does not map $F\langle t;\, s\rangle$ onto itself.*

PROOF. Since s is not the identity automorphism of F, there exists an element d of F such that $ds \neq d$. This, together with $td=(ds)t$, implies that d does not belong to the center of $F\langle t;\, s\rangle$. Hence the mapping

$$(d): x \to dx - xd, \qquad x \in F\langle t;\, s\rangle \tag{6}$$

is a nonzero inner-derivation of $F\langle t;\, s\rangle$. Now if (d) is to map $F\langle t;\, s\rangle$ onto itself, then in particular the equation

$$dx - xd = c_0 t^0 \tag{7}$$

must have a solution $x=\sum_{i>n}^{\infty} x_i t^i$ for each nonzero choice of c_0 in F. But (7) is equivalent to the set of equations

$$\begin{aligned} x_i(d - ds^i) &= 0 && \text{for } i \neq 0,\\ &= c_0 && \text{for } i = 0, \end{aligned}$$

which obviously can have no solution for x_0 for a nonzero choice of c_0. Consequently, the nonzero inner-derivation (6) does not map $F\langle t;\, s\rangle$ onto itself. This completes the proof.

COROLLARY. *Let F be a field and s an automorphism of F other than the identity. Then the Hilbert division ring $F\langle t;\, s\rangle$ does not satisfy condition* (C).

PROOF. This corollary follows immediately from Theorems 3 and 4.

P. M. Cohn has shown in [1] that an algebra R without zero divisors or unit-element can be embedded in an algebra S (over the same field as R) such that the equation $ax-xb=c$ has a solution in S for any a, b, c in S such that $a\neq 0$ and $b\neq 0$. It would seem that perhaps a similar embedding could be carried out to give an affirmative answer to our Question 1. However this does not seem possible, at least along the lines of [1], for the following reasons. First of all one cannot hope to embed an arbitrary algebra R without zero divisors (with or without unit-element) in an algebra S satisfying (C), for (C)

would imply that S is a division ring and Malcev [6] has given an example of an algebra without zero-divisors that can not be embedded in a division ring. On the other hand if one starts with a division ring Δ and tries to apply Cohn's methods in [1] to embed it in a division ring Δ^* satisfying (C), then the following difficulty arises. Cohn's arguments are for 1-algebras without zero-divisors. However, no division ring is a 1-algebra and the 1-algebra of a division ring always has zero divisors. Similarly, Cohn's paper [2] does not seem to be related to our questions on division rings.

Added in proof. It has been pointed out to the author by Mr. E. E. Lazerson that the following result, more general than Theorem 2, is just as easily established as Theorem 2. *If* Δ *satisfies* (C) *then each element of* $\Delta - F$ *is transcendental.* Just recently, Mr. Lazerson has succeeded in giving an affirmative answer to the author's Question 2. His work is to appear in the Bulletin of the American Mathematical Society. Questions 1 and 3 remain open.

References

1. P. M. Cohn, *On a class of simple rings*, Mathematika vol. 5 (1958) pp. 103–117.

2. ———, *Simple rings without zero-divisors and Lie division rings*, Mathematika vol. 6 (1959) pp. 14–18.

3. Bruno Harris, *Commutators in division rings*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 628–630.

4. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloquium Publications, vol. 37, 1956.

5. R. E. Johnson, *On the equation* $x\alpha = \gamma x + \beta$ *over an algebraic division ring*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 202–207.

6. A. Malcev, *On the immersion of an algebraic ring into a field*, Math. Ann. vol. 113 (1936–1937) pp. 686–691.

7. Hans J. Zassenhaus, *The theory of groups*, 2nd ed., New York, Chelsea, 1958.

University of Nebraska