A PROOF OF A FUNCTIONAL EQUATION RELATED TO THE THEORY OF PARTITIONS

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Introduction. The purpose of this note is to give a simple alternative proof for the following functional equation:

$$\begin{aligned}\sum_{l=0}^{\infty}\{\lambda((l+\alpha)z-i\beta)+\lambda((l+1-\alpha)z+i\beta)\}+\pi z(\alpha^2-\alpha+1/6)\\ =\sum_{l=0}^{\infty}\{\lambda((l+\beta)z^{-1}+i\alpha)+\lambda((l+1-\beta)z^{-1}-i\alpha)\}\\ +\pi z^{-1}(\beta^2-\beta+1/6)+2\pi i(\alpha-1/2)(\beta-1/2),\end{aligned} \tag{1}$$

where $\lambda(t)=-\log(1-e^{-2\pi t})$ (the principal value), $0\leqq\alpha\leqq 1$, $0<\beta<1$ (or $0<\alpha<1$, $0\leqq\beta\leqq 1$), and $\Re(z)>0$. This formula was first proved in [**2**], and its applications to the theory of partitions have also been investigated (see [**2**; **3**]).

The method we employ in this paper is essentially that of Rademacher [**4**]. Although our method is still based on the Mellin transform technique and the theory of the Hurwitz zeta-functions, we have made a thorough revision of Rademacher's original method, and thus a very short and direct proof of the functional equation can be given.

It may be noted that if we define a function $\Lambda(z, \alpha, \beta)$ by

$$\begin{aligned}\Lambda(z,\alpha,\beta)=\sum_{l=0}^{\infty}\{\lambda((l+\alpha)z-i\beta)+\lambda((l+1-\alpha)z+i\beta)\}\\ +\pi z(\alpha^2-\alpha+1/6)-\pi i(\alpha-1/2)(\beta-1/2),\end{aligned}$$

then (1) is written in the form

$$\Lambda(z,\alpha,\beta)=\Lambda(z^{-1},1-\beta,\alpha).$$

On the other hand, it is clear from the definition that

$$\Lambda(z,\alpha,\beta)=\Lambda(z,1-\alpha,1-\beta).$$

Proof of the functional equation. To prove (1), we first assume that $0<\alpha<1$, $0<\beta<1$, $\Re(z)>0$. Using the expansion

$$\lambda(x)=-\log(1-e^{-2\pi x})=\sum_{n=1}^{\infty}n^{-1}e^{-2\pi xn}$$

and applying the Mellin formula

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$$e^{-y} = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Gamma(s) y^{-s} ds \qquad (\Re(y) > 0)$$

with $c = 3/2$, it follows that

$$\sum_{l=0}^{\infty} \lambda((l+\alpha)z - i\beta) = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} e^{-2\pi((l+\alpha)z - i\beta)n}$$

$$= \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \frac{e^{2\pi i\beta n}}{2\pi i n} \int_{3/2-i\infty}^{3/2+i\infty} \Gamma(s) \{2\pi(l+\alpha)zn\}^{-s} ds$$

$$= \frac{1}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} \frac{\Gamma(s)}{(2\pi z)^s} \left(\sum_{l=0}^{\infty} \frac{1}{(l+\alpha)^s} \right) \left(\sum_{n=1}^{\infty} \frac{e^{2\pi i\beta n}}{n^{1+s}} \right) ds.$$

Therefore

$$(2) \qquad \sum_{l=0}^{\infty} \lambda((l+\alpha)z - i\beta) = \frac{1}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} \frac{\Gamma(s)}{(2\pi z)^s} \zeta(s, \alpha) \zeta_\beta(1+s) ds,$$

where $\zeta(s, \alpha) = \sum_{l=0}^{\infty} (l+\alpha)^{-s}$ is the Hurwitz zeta-function, and $\zeta_\beta(s) = \sum_{n=1}^{\infty} n^{-s} e^{2\pi i\beta n}$ $(\Re(s) > 0)$.

Now the function $\zeta_a(s)$ $(0 < a < 1)$ can be expressed in terms of Hurwitz zeta-functions, i.e.,

$$(3) \qquad \zeta_a(s) = \Gamma(1-s)(2\pi)^{s-1} \{ e^{\pi i(1-s)/2} \zeta(1-s, a) + e^{\pi i(s-1)/2} \zeta(1-s, 1-a) \}.$$

This relation is an easy consequence of the famous formula:

$$\zeta(s, a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ \sin(\pi s/2) \sum_{n=1}^{\infty} \frac{\cos(2\pi a n)}{n^{1-s}} + \cos(\pi s/2) \sum_{n=1}^{\infty} \frac{\sin(2\pi a n)}{n^{1-s}} \right\} \qquad (\Re(s) < 0).$$

From (3) we see that $\zeta_a(s)$ is an integral function of s (cf. Apostol [1]).

Returning to (2), we obtain further that

$$(4) \qquad \sum_{l=0}^{\infty} \{ \lambda((l+\alpha)z - i\beta) + \lambda((l+1-\alpha)z + i\beta) \} = \frac{1}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} z^{-s} F(s; \alpha, \beta) ds$$

with

$$(5)\quad F(s;\alpha,\beta) = \Gamma(s)(2\pi)^{-s}\{\zeta(s,\alpha)\zeta_\beta(1+s) + \zeta(s,1-\alpha)\zeta_{1-\beta}(1+s)\}.$$

The function $F(s;\alpha,\beta)$ satisfies the equation

$$(6)\qquad F(s;\alpha,\beta) = F(-s;1-\beta,\alpha),$$

since we have, by (5) and (3),

$$(7)\quad \begin{aligned} &F(s;\alpha,\beta)\\ &= \Gamma(s)\Gamma(-s)[e^{-\pi is/2}\{\zeta(s,\alpha)\zeta(-s,\beta) + \zeta(s,1-\alpha)\zeta(-s,1-\beta)\}\\ &\quad + e^{\pi is/2}\{\zeta(s,\alpha)\zeta(-s,1-\beta) + \zeta(s,1-\alpha)\zeta(-s,\beta)\}],\end{aligned}$$

which is unchanged when s, α, β are replaced by $-s$, $1-\beta$, α respectively.

Now in the integral on the right of (4), we move the path of integration from $\Re(s)=3/2$ to $\Re(s)=-3/2$, and obtain

$$(8)\quad \begin{aligned} &\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} z^{-s}F(s;\alpha,\beta)ds\\ &\qquad = \frac{1}{2\pi i}\int_{-3/2-i\infty}^{-3/2+i\infty} z^{-s}F(s;\alpha,\beta)ds + R_1 + R_0 + R_{-1},\end{aligned}$$

where R_1, R_0, R_{-1} are the corresponding residues of the integrand at its simple poles $s=1$, 0, -1 respectively. The displacement of the path of integration may easily be justified by considering the order of magnitude of the integrand. Actually, we have, writing $\Im(s)=t$,

$$\begin{aligned} z^{-s} &= O(e^{(\pi/2-\delta)|t|}) \quad \text{for } |\arg z| \leqq \pi/2-\delta < \pi/2,\\ \Gamma(s)\Gamma(-s) &= -\pi(s\sin\pi s)^{-1} = O(|t|^{-1}e^{-\pi|t|}),\\ \zeta(s,a) &= O(|t|^c) \qquad\qquad \text{(see [5, p. 276])},\end{aligned}$$

and hence, by (7),

$$z^{-s}F(s;\alpha,\beta) = O(|t|^{2c-1}e^{-\delta|t|}).$$

The values of the residues are calculated as follows:

$$\begin{aligned} R_1 &= (2\pi z)^{-1}\{\zeta_\beta(2) + \zeta_{1-\beta}(2)\}\\ &= (2\pi z)^{-1}\sum_{n=1}^{\infty} n^{-2}(e^{2\pi i\beta n} + e^{-2\pi i\beta n})\\ &= (\pi z)^{-1}\sum_{n=1}^{\infty} n^{-2}\cos(2\pi\beta n) = \pi z^{-1}(\beta^2-\beta+1/6),\end{aligned}$$

$$\begin{aligned} R_0 &= \zeta(0, \alpha)\zeta_\beta(1) + \zeta(0, 1-\alpha)\zeta_{1-\beta}(1) \\ &= (1/2 - \alpha) \sum_{n=1}^{\infty} n^{-1}(e^{2\pi i\beta n} - e^{-2\pi i\beta n}) \\ &= 2i(1/2 - \alpha) \sum_{n=1}^{\infty} n^{-1} \sin(2\pi\beta n) \\ &= 2\pi i(1/2 - \alpha)(1/2 - \beta), \end{aligned}$$

and, using (6),

$$R_{-1} = -\pi z(\alpha^2 - \alpha + 1/6).$$

Inserting these values and the relation (6) into the right member of (8), changing the variable s to $-s$ and using (4), it is found that the required equation (1) holds for $0<\alpha<1$, $0<\beta<1$, and $\Re(z)>0$. The validity of (1) for the end-points of the interval of α or β may then be established by means of letting $\alpha\to+0$, $1-0$ or $\beta\to+0$, $1-0$ in (1). This completes the proof of the functional equation.

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