CARTWRIGHT'S THEOREM ON FUNCTIONS BOUNDED AT THE INTEGERS

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1. The theorem of Cartwright [4] that for functions of exponential type not exceeding $k<\pi$ bounded at the integers

$$\sup_{-\infty<x<+\infty} |f(x)| \leqq A(k)M, \quad M = \sup |f(n)|, \quad n = 0, \pm 1, \pm 2, \cdots$$

has been followed by many estimates of the value of $A(k)$ [1; 3; 5; 6]. We shall generally mean by $A(k)$ its smallest possible value. It has been shown [3] that as k approaches π, $A(k)$ must tend to infinity like $\log\{1/(\pi-k)\}$. For k near π the estimates of $A(k)$ from above and below are in close agreement [5]. But for small k, for example $0<k<\pi/2$ we have only [1; 3]

$$1 \leqq A(k) \leqq 2 + \frac{\pi}{3(\pi - k)}.$$

The lower estimate can be improved. The function $f(x)=\sin(\pi x/N)$ is of type π/N and if N is an odd integer $\sup|f(n)| = \cos(\pi/2N)$ while $\sup|f(x)| = 1$. At any rate for some small values of k we have

$$A(k) \geqq \sec(k/2). \tag{1}$$

It is a very natural conjecture that $A(k)$ tends to unity as k tends to zero.

We are able to establish this conjecture in the following way.

Given the existence of $A(k)$ and an upper estimate $A_0(k)$ it follows from Bernstein's theorem [2, p. 206] that

$$|f'(x)| \leqq kA_0(k)M, \qquad |f''(x)| \leqq k^2A_0(k)M, \cdots. \tag{2}$$

Use of these inequalities in various ways leads to different estimates for $A(k)$, the best being

$$A(k) \leqq (1 - k^2/8)^{-1}, \qquad (0 < k < 2^{3/2}), \tag{3}$$

$$A(k) \leqq 2/(3 - k), \qquad (2 < k < 3). \tag{4}$$

Comparison with (1) shows that for small k our estimate (3) is asymptotically correct. The upper and lower estimates are each $1+k^2/8+O(k^4)$. For $k=\pi/2$ we have $A(\pi/2)\geqq 2^{1/2}$, which is also ob-

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tained in [4]. The estimate (3) is numerically 1.446 which exceeds $2^{1/2}$ by less than 3 per cent.

2. If the inequality (2) is integrated between x and the nearest integer n we have

$$(5) \qquad |f(x) - f(n)| \leqq |x - n| kA_0(k)M \leqq kA_0(k)M/2$$

and hence

$$|f(x)| \leqq \{1 + kA_0(k)/2\}M.$$

The new constant $1+kA_0(k)/2$ will be less than $A_0(k)$ if $k<2$ and

$$A_0(k) > 1 + kA(k)/2 \quad \text{or} \quad A_0(k) > 2/(2-k).$$

We can infer that

$$|f(x)| \leqq 2M/(2-k)$$

by an iterative argument. Set

$$A_1(k) = 1 + kA_0(k)/2, \qquad A_{n+1}(k) = 1 + kA_n(k)/2.$$

Evidently

$$|f(x)| \leqq A_n(k)M = \{2/(2-k) + (k/2)^n[A_0(k) - 2/(2-k)]\}M.$$

Since n can be arbitrarily large,

$$|f(x)| \leqq 2M/(2-k).$$

3. This simple argument is sufficient to establish the conjecture that $A(k)$ tends to unity as k tends zero.[1] A slight improvement is obtained by using a variant of Bernstein's theorem [2, p. 214] namely that if $f(z)$ is an entire function of exponential type k bounded on the real axis then for $0<2\delta<\pi/k$

$$|f(t+\delta) - f(t-\delta)| \leqq 2\sin(\delta k) \sup |f(x)|.$$

If this inequality is used with $x=t\pm\delta$ and $t\mp\delta$ the nearest integer, in place of (5) then $\delta\leqq 1/4$ and

$$\sup |f(x)| \leqq M + 2\sin(k/4)A_0(k)M.$$

Arguing as before we now infer that

$$\sup |f(x)| \leqq \{1 - 2\sin(k/4)\}^{-1} \sup |f(x)|,$$

this inequality being valid for $0<k<2\pi/3$.

[1] R. P. Boas, Jr. gives us another proof. Suppose $pk<\pi$, p is an integer; then $f(z)^p$ is of type pk and $|f(x)|^p \leqq M^p$. So $|f(x)|^p \leqq A(pk)M^p$, $|f(x)| \leqq A(pk)^{1/p}M$.

4. Lagrange's interpolation formula [7]

$$f(x) = \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b) + \lambda \frac{(x-a)(x-b)}{2} f''(\xi), \quad |\lambda| \leqq 1, \tag{6}$$

may also be used with $a = n < x < b = n+1$.

Since $|f''(x)| \leqq k^2 A_0(k) M$ by repeated use of Bernstein's theorem, we have

$$|f(x)| \leqq M + (x-a)(b-x)k^2 A_0(k) M/2 \leqq M + k^2 A_0(k) M/8.$$

This leads to the estimate

$$\sup |f(x)| \leqq (1 - k^2/8)^{-1} \sup |f(n)|$$

valid for $k < 2^{3/2}$.

It may be noted that if $k > 2$ our use of Lagrange's interpolation is inferior to the more elementary inequality

$$|f(x) - f(n)| \leqq |x - n| \, k A_0(k) M \tag{7}$$

when $|x-n| < 1/2 - 1/k$. If we use (7) in the intervals $n \leqq x \leqq n + 1/2 - 1/k$, $n+1-(1/2-1/k) \leqq x \leqq nx+1$ and (6) for $a \leqq x \leqq b$ with $a = n + 1/2 - 1/k$ and $b = n+1-(1/2-1/k)$, we evidently obtain

$$|f(x)| \leqq M + (1/2 - 1/k) k A_0(k) M + k^{-2} k^2 A_0(k) M/2.$$

This leads to the inequality

$$\sup |f(x)| \leqq \frac{2}{3-k} \sup |f(n)|$$

valid for $2 < k < 3$.

References

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