A THEOREM ON OVERCONVERGENCE

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The conjecture announced by A. J. Macintyre [2; 3] is equivalent to the theorem stated and proved below.

Theorem. Let $D$ be an open domain containing the origin and let $f(z)$ be a function regular in $D$ with the expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Let $D_1$ be a bounded closed domain contained in $D$. Then there exists a positive number $\omega = \omega(D, D_1)$ such that if $c_n = 0$ for a sequence of intervals $n_k \leq n \leq \lambda n_k$ with $\lambda > \lambda_0$, then the subsequence of partial sums $s_{n_k} = \sum_{n=0}^{n_k} c_n z^n$ converges uniformly to $f(z)$ in $D_1$.

Proof. Let $CD$ and $CD_1$ denote the complements of $D$ and $D_1$ respectively and let $h_i, i=1, 2, \cdots$, be the components of $CD_1$. The components can be considered as disjoint and there exists only one unbounded component. The one unbounded component will be denoted as $h_1$.

One can assert that there exists only a finite number of components $h_i$ such that

(1) $h_i \cap CD \neq \emptyset$,

where $\emptyset$ is the empty set. This assertion is proved as follows. Assume that there exists an infinite number of components $h_i, i \geq 2$, such that (1) is valid. A bounded sequence of points $a_i$ can be formed where $a_i \in h_i \cap CD, i \geq 2$. Every $a_i$ is an element of $CD$ and hence the dis-

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tance \( d \) from \( D_1 \) is at least \( \delta > 0 \). The limit point \( a \) of the sequence then must be such that \( d(a, D_1) \geq \delta > 0 \). Thus \( a \) is an element of \( CD_1 \) and all points \( z \) in \( |z - a| < \delta \) must be in the same component.

Let the finite number of components be enumerated as \( h_i, i = 1, 2, \cdots, N \). Considering now

\[
D_2 = D_1 + \bigcup_{i=N+1}^{\infty} h_i,
\]

then \( D_2 \) is a bounded closed domain and \( D_2 \subset D_1 \). Since \( \sum_{i=N+1}^{\infty} h_i \) is bounded and \( D_1 \) is bounded by hypothesis, \( D_2 \) is bounded. Also, since \( h_i \cap CD = \emptyset, i \geq N+1 \), then \( h_i \subset D \) and \( D_2 \) is contained in \( D \). To prove that \( D_2 \) is closed note that its complement is \( \sum_{i=1}^{N} h_i \) and is open.

Now \( N-1 \) polygonal arcs \( L_1, L_2, \cdots, L_{N-1} \) can be chosen such that \( D_2 - \sum_{i=1}^{N-1} L_k \) is simply connected. Also \( N-1 \) other polygon arcs \( L_1', L_2', \cdots, L_{N-1}' \) can be so chosen that \( L_k \cap L_{k}' = \emptyset \) and \( D_2 - \sum_{i=1}^{N-1} L_k' \) is simply connected. Consider now the open circle \( C(s, R) \) or \( |z - s| < R \) and let \( S(L, R) = \bigcup_{z \in L} C(s, R) \). Thus \( S(L, R) \) is a strip enclosing the polygonal arc \( L \). For \( R \) sufficiently small,

\[
S(L_k, R) \cap S(L_j', R) = \emptyset.
\]

Hence for \( R \) sufficiently small two closed simply connected domains can be defined, \( D_3 = D_2 - \bigcup_{i=1}^{N-1} S(L_k, R) \) and \( D_3' = D_2 - \bigcup_{i=1}^{N-1} S(L_j', R) \) such that \( D_3 + D_3' = D_2 \). This follows from

\[
D_3 + D_3' = D_2 - \left\{ \bigcup_{k=1}^{N-1} S(L_k, R) \right\} \cap \left\{ \bigcup_{j=1}^{N-1} S(L_j', R) \right\} = D_2
\]

by (3).

The proof of the theorem follows. An open bounded simply connected domain \( \Delta = \Delta(D, D_3) \) can now be defined such that \( D_3 \subset \Delta \), \( \{ |z| < r \} \subset \Delta, \Delta \subset D \) where \( r \) is the radius of convergence of \( f(z) = \sum_{n=0}^{\infty} c_n z^n \) and \( \Delta \) is the closure of \( \Delta \). From the Nevanlinna two-constant theorem, if \( F(z) \) is regular in \( \Delta \)

\[
M(\Delta) = \mathrm{l.u.b.} \frac{1}{|z|}, \quad M(d) = \mathrm{l.u.b.} \left| F(z) \right|, \quad |z| < r/2
\]

then \([1]\)

\[
M(D_3) = \mathrm{l.u.b.} \left| F(z) \right| \leq \left\{ M(\Delta) \right\}^\theta \left\{ M(d) \right\}^{1-\theta}
\]

where \( \theta > 0 \) depends on \( D_3 \) and \( \Delta \). Using the majorization of \( r_{n_k} \), where \( r_{n_k} = f(z) - s_{n_k}, r_{n_k} = \sum_{n=k}^{\infty} c_{n} z^n \), if \( n_k \) is large we get \( \mathrm{l.u.b.} \left| \right| \left| r_{n_k} \right| < \)
$H(3/4)^{\lambda n_k}$ and $\limsup_{n \to \Delta} |r_{n_k}| < H_1^{n_k}$ where $H$ and $H_1$ are two constants depending only on $\Delta$. Thus by (4),

$$\limsup_{z \in D_3} |r_{n_k}| \leq H_1^{1-\theta} \left\{ H_1^{\theta} (3/4)^{\lambda (1-\theta)} \right\}^{n_k}.$$

Thus if $\lambda > \lambda_0(\Delta, D_3)$ there is overconvergence in $D_3$. Similarly there is overconvergence in $D'_3$ if $\lambda > \lambda_0(\Delta, D'_3)$. Now since $D_3 + D'_3 = D_2 \supset D_3$ the theorem is proved.

**Remark.** By the same method similar results are proved for the series of Dirichlet and for the integral of Laplace.

**References**


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