ON RELATIVELY NONATOMIC MEASURES

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1. **Introduction and summary.** Let Ω be a set, $\mathcal{A}$ a σ-algebra of subsets of Ω, and m be a measure defined on $\mathcal{A}$ whose range R is a subset of k-dimensional Euclidean space E_k, i.e., m is a completely additive set function defined on $\mathcal{A}$ such that each component of m is nonnegative. If each component of m is nonatomic it was shown by Lyapunov [3], Halmos [2], and Blackwell [1] that R is a convex set. Now let $\mathcal{A}$ be the class of Borel subsets of the real line, let $m=(m_1, m_2)$ where m_1 is Lebesgue measure and $m_2(A)$ counts the number of integers in the set A. Then R is not convex, in fact R is the set of points $\{(x, y)\}$ where $x\geqq 0$ and y is a nonnegative integer. Yet if $(x, y)\in R$ and (x, y) lies on the line segment connecting the zero vector and $m(A)$ it is easily seen that there exists $A'\subset A$ such that $m(A')=(x, y)$. In this note we give a definition of relative nonatomicity which covers situations of this kind and prove the analogue of the convexity theorem.

2. **Definitions and results.** We shall assume throughout that every subset of Ω that is discussed is measurable, i.e., an element of $\mathcal{A}$.

DEFINITION 1. Let R be the range of m. m is nonatomic relative to R if for every set A and every number α with $0<\alpha<1$ such that $\alpha m(A) \in R$ there exists $A'\subset A$ and a number α' with $0<\alpha'<1$ such that $m(A')=\alpha' m(A)$.

DEFINITION 2. Let A be a set. We define $R(A)=\{r\in R \mid r=\alpha m(A)$ with $0<\alpha\leqq 1\}$. Let $\alpha_0=\inf\{\alpha \mid \alpha>0,\ \alpha m(A)\in R\}$. Then we define $r_0(A)=\alpha_0 m(A)$.

THEOREM. *Let m be nonatomic relative to R. Let A be a set and $r\in R(A)$. Then there exists $B\subset A$ with $r=m(B)$.*

The proof of the theorem will proceed by way of several lemmas.

LEMMA 1. *In Definition 1 we may choose A' and α' such that $\alpha'\leqq\alpha$.*

PROOF. Suppose the conclusion of the lemma is false. Let $A'\subset A$ such that $m(A')=\alpha' m(A)$ with $0<\alpha'<1$. Then $m(A-A')=(1-\alpha')m(A)$ and $\alpha'>\alpha$, $1-\alpha'>\alpha$. Hence $\alpha<1/2$. Now $r=\alpha m(A)=(\alpha/\alpha')\alpha' m(A)=(\alpha/\alpha')m(A')$, and similarly

$$r = [\alpha/(1-\alpha')]m(A-A').$$

Hence we may apply the definition and the above procedure separately to A' and $A-A'$. After some manipulation we obtain $\alpha<1/4$.

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Repeating this process indefinitely yields $\alpha = 0$, which is a contradiction.

LEMMA 2. *Suppose m is nonatomic relative to R. Then for every set A there exists $B \subset A$ such that $m(B) = r_0(A) = \alpha m(A)$ for some α, with $0 \leqq \alpha \leqq 1$.*

PROOF. If $r_0(A)$ is the null vector choose B to be the empty set and $\alpha = 0$, and if $r_0(A) = m(A)$ choose $B = A$ and $\alpha = 1$. If $r_0(A) \in R(A)$ the conclusion follows from Lemma 1. Otherwise there exists a strictly decreasing sequence of numbers $\{\alpha_n\}$ with $0 < \alpha_n < 1$ such that $r_0(A) = \lim_n \alpha_n m(A)$, and such that $\alpha_n m(A) \in R(A)$. But then it follows again from Lemma 1 that there exists a sequence of number $\{\alpha_n'\}$ and a sequence of sets $\{B_n\}$ which may be chosen so that $A \supset B_1 \supset B_2 \supset \cdots$ such that $\lim_n m(B_n) = \lim_n \alpha_n' m(A) = r_0(A)$. If $B = \lim_n B_n$ then clearly B satisfies the conclusion of the lemma.

LEMMA 3. *Suppose m is nonatomic relative to R. Let A be a set and suppose $B \subset A$ satisfies the conclusion of Lemma 2, i.e., $m(B) = r_0(A)$. Then either $m(B)$ is the null vector or $m(A)$ is an integral multiple of $m(B)$.*

PROOF. Suppose $m(B)$ is not the null vector, i.e., $m(B) = \alpha m(A)$ for some α with $0 < \alpha \leqq 1$. If $\alpha = 1$ we are done. Assume then that $0 < \alpha < 1$. Choose $B' \subset A - B$ so as to satisfy Lemma 2, i.e., $m(B') = r_0(A - B) = \alpha' m(A - B) = \alpha'(1 - \alpha) m(A)$ for some α' with $0 \leqq \alpha' \leqq 1$. From the definition of the function r_0 it follows that $\alpha'(1 - \alpha) \geqq \alpha$. If $\alpha < \alpha'(1 - \alpha)$ we may write $m(B) = [\alpha / (\alpha'(1 - \alpha))] m(B')$. It then follows from the definition of relative nonatomicity that there exist $B'' \subset B'$ and α'' with $0 < \alpha'' < 1$ such that $m(B'') = \alpha'' m(B')$. But this contradicts the fact that B' is minimal for $A - B$, i.e., $m(B') = r_0(A - B)$. Hence $m(B') = m(B)$. Now if $m(A - B - B')$ is the null vector we are done. Otherwise we repeat this process. But clearly this must stop in a finite number of steps, and the lemma is proved. By the same techniques we have immediately

LEMMA 4. *Suppose m is nonatomic relative to R. Let A be a set and suppose $r_0(A)$ is not the null vector. Then A is the union of finitely many disjoint sets, each having measure $r_0(A)$.*

LEMMA 5. *Suppose m is nonatomic relative to R. Let A be a set and suppose $r_0(A)$ is not the null vector. Then every $r \in R(A)$ is a positive integral multiple of $r_0(A)$.*

PROOF. Let $r \in R(A)$, i.e., $r = \alpha m(A)$ with $0 < \alpha \leqq 1$. From Lemma 3 it follows that there exists $B \subset A$ such that $m(A) = nm(B) = nr_0(A)$ for some positive integer n. If $\alpha = 1$ there is nothing to prove. Assume then that $0 < \alpha < 1$ and that $\alpha = (k+c)/n$ where k is an integer with $1 \leqq k < n$ and c is a number with $0 < c < 1$. The case $k = 0$ is impossible for in that case $\alpha < 1/n$. Now $r \in R$ and hence there exists a set C such that $r = m(C)$. Now consider $r_0(C) = \alpha' m(C) = \alpha' \alpha m(A)$ for some α' with $0 \leqq \alpha' \leqq 1$. If $\alpha' = 0$ then $r_0(A)$ is the null vector which is contrary to the hypothesis. Consequently $\alpha' > 0$ and $m(C) = im(C_1)$ for some set $C_1 \subset C$ and some positive integer i, from Lemma 3. Now $m(B) = (1/n)m(A) = (1/(k+c))m(C)$ and hence $(1/i) \leqq (1/(k+c))$. Since $k+c$ is not an integer we have $k+c < i$. But then $m(C_1) = [(k+c)/(in)]m(A)$ and $(k+c)/(in) < 1/n$ which contradicts the minimality of $1/n$. Hence $k+c$ must be an integer and the lemma is proved.

Proof of the theorem. If $r_0(A)$ is not the null vector the conclusion of the theorem follows at once from Lemma 4 and Lemma 5. Suppose then that $r_0(A)$ is the null vector. Now $r = \alpha m(A)$ for some α with $0 \leqq \alpha \leqq 1$. If $\alpha = 0$ or $\alpha = 1$ the conclusion is trivial. Assume then that $0 < \alpha < 1$. Let $\mathfrak{F} = \{B \subset A \mid m(B) = \beta m(A) \text{ with } 0 < \beta \leqq \alpha\}$. We partially order $\mathfrak{F}$ by saying that $B_1 < B_2$ if $B_1 \subset B_2$ and if the corresponding β_1 and β_2 satisfy $\beta_1 < \beta_2$. If $\mathfrak{F}'$ is a linearly ordered subfamily of $\mathfrak{F}$ it is easily seen that $\mathfrak{F}'$ has an upper bound in $\mathfrak{F}$. Consequently Zorn's lemma applies. Let B be a maximal element of $\mathfrak{F}$ and suppose $m(B) = \beta m(A)$. We shall show that $\beta = \alpha$. Suppose $\beta < \alpha$. Since $r_0(A)$ is the null vector it follows easily that $r_0(A - B)$ is the null vector. Hence we can find an arbitrarily small positive number γ and a corresponding set $B' \subset A - B$ such that $m(B') = \gamma m(A - B) = \gamma(1-\beta)m(A)$. Let $B'' = B \cup B'$. Then $m(B'') = [\beta + \gamma(1-\beta)]m(A)$ and by choosing γ sufficiently small we violate the maximality of B. Thus $\beta = \alpha$ and theorem is proved.

References

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