APPROXIMATING COMPACT AND WEAKLY COMPACT OPERATORS[1]

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THEOREM 1. *A linear operator T is compact if and only if it is a cluster point for the topology α of a sequence $\{F_n\}$ of continuous linear operators with finite dimensional range.*

REMARKS. To place the theorem in context consider all operators as mapping the Banach space X into the Banach space Y and consider the topology α to be that of almost uniform convergence on the unit ball of X utilizing the norm topology on Y. The second theorem gives a similar result for weakly compact operators. All definitions and background information can be found in the two references.

PROOF OF THEOREM 1. Assuming the sequence $\{F_n\}$ to have T as a cluster point, there is a subnet $\{F_\gamma, \gamma\in G\}$ converging to T for the topology α. The second adjoint operators $\{F_\gamma^{**}, \gamma\in G\}$ form a Cauchy net for the topology of almost uniform convergence on the unit ball S^{**} of X^{**} utilizing the norm topology on Y^{**} [1, Theorem 4.1]. Define a linear operator F_0 which agrees with T^{**} on the image of X, while for all other x^{**} in X^{**}, $F_0(x^{**})=\lim_\gamma F_\gamma^{**}(x^{**})$.

Consider an arbitrary positive number ϵ and a net $\{x_\delta^{**}, \delta\in D\}$ in the image of S in X^{**} converging to a point x_0^{**} in X^{**} for the X^* topology. There exist δ_0 in D and $\gamma_1, \gamma_2, \cdots, \gamma_k$ in G such that

$$\min_{i=1,2,\ldots,k} \|F_0(x_\delta^{**}) - F_{\gamma_i}^{**}(x_\delta^{**})\| < \frac{\epsilon}{3} \qquad \text{for all } \delta > \delta_0,$$

$$\|F_{\gamma_i}^{**}(x_\delta^{**}) - F_{\gamma_i}^{**}(x_0^{**})\| < \frac{\epsilon}{3}$$

for $i=1, 2, \cdots, k$ and all $\delta>\delta_0$, and $\|F_{\gamma_i}^{**}(x_0^{**})-F_0(x_0^{**})\|<\epsilon/3$ for $i=1, 2, \cdots, k$. Thus $\|F_0(x_\delta^{**})-F_0(x_0^{**})\|<\epsilon$ for all $\delta>\delta_0$ and F_0 is continuous for the X^* topology on S^{**} and the norm topology on Y^{**}. Therefore F_0 is T^{**} and T is compact.

For the converse let $\mathcal{P}$ be the directed set composed of all continuous projections with finite dimensional range in Y, the order being determined by the inclusion ordering on their ranges. The net $\{P^{**}T^{**}, P\in\mathcal{P}\}$ converges pointwise to T^{**} on S^{**} and a known theorem for continuous functions says that due to the metric topology on Y^{**} and the compactness of S^{**} the net can be replaced by a

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sequence $\{P_n^{**}T^{**}\}$ having T^{**} as a cluster point.[2] T^{**} must also be a cluster point of the sequence for the topology of almost uniform convergence on S^{**} with the norm topology on Y^{**} [1, Theorem 4.2]. Therefore it is concluded that T is a cluster point of the sequence $\{P_nT\}$ for the topology α.

By omitting the sequence and resorting to the topology β, almost uniform convergence on the unit ball of X with the weak topology on Y, the same line of reasoning gives Theorem 2.

THEOREM 2. *A linear operator T is weakly compact if and only if it is the limit point for the topology β of a net $\{F_\gamma, \gamma \in G\}$ of continuous linear operators with finite dimensional range.*

REFERENCES

1. J. W. Brace, *The topology of almost uniform convergence*, Pacific J. Math. vol. 9 (1959) pp. 643–652.

2. N. Dunford and J. T. Schwartz, *Linear operators, Part* I, New York, Interscience, 1958.

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[2] The author is unable to locate a reference. A version of the theorem appeared at one time in a manuscript copy of J. L. Kelley's book on linear topological spaces. The statement is as follows. "Let F be a subset of the space of continuous functions with compact domain S and range in a metric space. If f is in the closure of F for the topology of pointwise convergence, then there is a sequence in F having f as a cluster point."