CONTINUOUS IMAGES OF BOREL SETS

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1. Introduction. It is well known that in a complete separable metric space every Borel set is the one-to-one continuous image of a $\mathcal{G}_d$ in some other such space and that the countable-to-one continuous image of a Borel set is Borel.

In a general topological space, if $K$ denotes the family of compact sets, G. Choquet [1] has pointed out that the proper families to consider in this context are Borelian $K$ instead of Borel $K$ and $K_{st}$ instead of $\mathcal{G}_d$ (see §2 for definitions). Let a space have property I if it is Hausdorff and the difference of two compact sets is a $K_*$. It has been shown by Choquet [1] that if $Y$ has property I then every Borelian $K$ set in $Y$ is the one-to-one continuous image of a $K_{st}$ in some compact Hausdorff $X$. On the other hand, in a previous paper [2] we proved that if $X$ has property I then the countable-to-one continuous image of a $K_{st}$ in $X$ to a compact Hausdorff space $Y$ is Borelian $K$ in $Y$.

In this paper we complete the picture. We first show that the difference of two compact sets is a $K_*$ iff it is analytic and conclude that a space has property I iff it is Hausdorff and Borel $K = $ Borelian $K$, thereby answering a question raised by Choquet [1, p. 139]. We then prove that if $X$ has property I then every Borel $K$ set in $X$ is the one-to-one continuous image of a $K_{st}$ in some $Y$, where $Y$ also has property I. Making use of our previous result, we conclude that the countable-to-one continuous image of a Borel $K$ set in $X$ to a compact Hausdorff space $Y$ is Borelian $K$ and it too has property I.

We are unable to determine whether the condition that $X$ have property I can be eliminated from the hypotheses, i.e., whether in any compact Hausdorff space $X$, every Borelian $K$ set is the one-to-one continuous image of a $K_{st}$ in some other compact Hausdorff space and whether the countable-to-one (or even one-to-one) image of a Borelian $K$ (or even a $K_{st}$) set in $X$ into a compact Hausdorff space is also Borelian $K$.

2. Notation and basic definitions.

2.1. $\omega$ denotes the set of all non-negative integers.

2.2. $K(X)$ is the family of all compact sets in $X$.

2.3. $K_*(X) = \{A : A = \bigcup_{i \in \omega} B_i \text{ for some sequence } B \text{ with } B_i \in K(X) \text{ for } i \in \omega \}$.

2.4. $K_{st}(X) = \{A : A = \bigcap_{i \in \omega} B_i \text{ for some sequence } B \text{ with } B_i \in K_*(X) \text{ for } i \in \omega \}$.

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2.5. Borel $K(X)$ is the smallest family $H$ such that $K(X) \subseteq H$ and if $A_i \in H$, for $i \in \omega$, then $\bigcup_{i \in \omega} A_i \in H$ and $A_0 - A_1 \in H$.

2.6. Borelian $K(X)$ is the smallest family $H$ such that $K(X) \subseteq H$ and if $A_i \in H$, for $i \in \omega$, then $\bigcup_{i \in \omega} A_i \in H$ and $\bigcap_{i \in \omega} A_i \in H$.

2.7. $A$ is analytic in $X$ iff $A$ is the continuous image of a $K_\sigma(X')$ for some Hausdorff space $X'$.

2.8. $X$ has property I iff $X$ is Hausdorff and, for every $A$ and $B$ in $K(X)$, $A - B \in K_\sigma(X)$.

2.9. $X$ has a countable compact base iff $X$ is Hausdorff and there exists a sequence $C$ with $C_i \in K(X)$, for $i \in \omega$, such that if $A$ is open in $X$ and $x \in A$ then, for some $i \in \omega$, $x \in C_i \subseteq A$.

2.10. $\prod_{i \in \omega} Y_i$ denotes the cartesian product of the $Y_i$, for $i \in \omega$.

2.11. The union topology for $\bigcup_{i \in \omega} Y_i$, where $Y_i \cap Y_j = \emptyset$ for $i \neq j$, is the topology in which $A$ is open iff $A = \bigcup_{i \in \omega} \alpha_i$, where $\alpha_i$ is open in $Y_i$ for $i \in \omega$.

3. Property I and countable compact bases. In this section, we study conditions under which a set has property I or a countable compact base. The main results are Theorems 3.1, 3.3, 3.5. The other results are needed in the next section.

3.1. **Theorem.** Let $X$ be Hausdorff, $A$ and $B$ in $K(X)$. Then $A - B$ is analytic in $X$ iff $A - B \in K_\sigma(X)$.

**Proof.** If $A - B \in K_\sigma(X)$ then clearly $A - B$ is analytic in $X$. Suppose now that $A - B$ is analytic in $X$. Then by Theorem 2.3 in [3], $A - B$ is Lindelöf in $X$, i.e., any open covering of $A - B$ can be reduced to a countable subcovering. Let

$$G = \{ \beta: \beta \text{ is open in } X \text{ and } B \subseteq \beta \},$$

$$F = \{ \alpha: \alpha = X - \text{closure } \beta \text{ for some } \beta \in G \}.$$

Since $X$ is Hausdorff, $F$ is an open covering of $X - B$ and hence a countable subfamily $F'$ covers $A - B$. Let $G'$ be a countable subfamily of $G$ such that

$$F' = \{ \alpha: \alpha = X - \text{closure } \beta \text{ for some } \beta \in G' \}$$

and let

$$H = \{ \gamma: \gamma = A - \beta \text{ for some } \beta \in G' \};$$

then $H$ is a countable family of compact sets whose union is $A - B$ so that $A - B \in K_\sigma(X)$.

3.2. **Lemma.** Borelian $K(X) \subseteq$ Borel $K(X)$.
3.3. Theorem. $X$ has property I iff $X$ is Hausdorff and Borelian $K(X) = \text{Borel } K(X)$.

Proof. Suppose $X$ has property I. Let $H$ be a maximal family such that $K(X) \subseteq H$ and if $A$ and $B$ are in $H$ then $A$ and $A - B$ are Borelian $K(X)$. Then we easily check that $H$ is closed under countable unions and intersections so that $H = \text{Borelian } K(X)$. Thus, if $A$ and $B$ are Borelian $K(X)$ then $A - B$ is also Borelian $K(X)$. Therefore Borel $K(X) \subseteq \text{Borelian } K(X)$ and in view of 3.2, Borel $K(X) = \text{Borelian } K(X)$.

Next, suppose $X$ is Hausdorff and Borelian $K(X) = \text{Borel } K(X)$. If $A$ and $B$ are in $K(X)$ then $A - B$ is Borelian $K(X)$ and hence (see e.g. [1, p. 142]) $A - B$ is analytic in $X$. Therefore, by 3.1, $A - B \in K_*(X)$.

3.4. Lemma. $X$ has property I iff $X$ is Hausdorff and for every $A \in K(X)$ and $B$ open in $X$ we have $A \cap B \in K_*(X)$.

3.5. Theorem. If $Y$ has a countable compact base and $X$ has property I then $X \times Y$ has property I.

Proof. Let $C$ be a sequence of compact sets in $Y$ such that if $U$ is open in $Y$ and $y \in U$ then, for some $i \in \omega$, $y \in C_i \subseteq U$. Suppose $A$ is compact and $B$ is open in $X \times Y$. Let, for each $i \in \omega$,

$$\beta_i = \{ x : \{ x \} \times C_i \subseteq B \}.$$ 

Then the $\beta_i$ are open in $X$. Moreover,

$$B = \bigcup_{i \in \omega} (\beta_i \times C_i)$$

for, if $(x, y) \in B$ then, for some $i \in \omega$,

$$y \in C_i \subseteq \{ z : (x, z) \in B \}$$

and hence $x \in \beta_i$ and $(x, y) \in \beta_i \times C_i$. Let $\alpha$ be the projection of $A$ onto $X$. Then $\alpha$ is compact in $X$ and $\alpha \cap \beta_i \in K_*(X)$ and hence

$$ (\alpha \cap \beta_i) \times C_i \subseteq K_*(X \times Y).$$

Since

$$A \cap B = \bigcup_{i \in \omega} (A \cap (\beta_i \times C_i)) = \bigcup_{i \in \omega} (A \cap ((\alpha \cap \beta_i) \times C_i))$$

we see that $A \cap B \in K_*(X \times Y)$. Therefore $X \times Y$ has property I.

3.5a. Corollary. If $Y$ is a metric space and $X$ has property I then $X \times Y$ has property I.
3.6. Theorem. If $X$ has property I, $C \subseteq K_{st}(X)$, $f$ is continuous on $C$, and $f(C)$ is Hausdorff then $f(C)$ has property I.

Proof. Let $A$ be compact and $B$ open in $f(C)$. Then $f^{-1}(A)$ is closed in $C$ and hence $f^{-1}(A) \subseteq K_{st}(X)$. Since $f^{-1}(B)$ is open in $C$ and $X$ has property I, we conclude

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \subseteq K_{st}(X).$$

Thus, $A \cap B$ is the continuous image of a $K_{st}(X)$, i.e., $A \cap B$ is analytic in $f(C)$, and hence, by 3.1, $A \cap B \subseteq K_{c}(f(C))$.

3.7. Theorem. Let $f$ be continuous and one-to-one on $C$ and $f(C)$ have property I. Then $C$ has property I.

Proof. Clearly $C$ must be Hausdorff. If $A$ and $B$ are compact in $C$, then $f(A)$ and $f(B)$ are compact in $f(C)$ and hence $f(A) - f(B) \subseteq K_{c}(f(C))$. Since $f$ is one-to-one, we have

$$A - B = A \cap f^{-1}(f(A) - f(B)) \subseteq K_{c}(C).$$

3.8. Lemma. If, for each $i \in \omega$, $Y_i$ is compact and has a countable compact base, then $\prod_{i \in \omega} Y_i$ is compact and has a countable compact base in the product topology.

3.9. Lemma. If, for each $i \in \omega$, $Y_i$ is compact and has a countable compact base and $Y_i \cap Y_j = \emptyset$ for $i \neq j$ then $\bigcup_{i \in \omega} Y_i$ is locally compact and has a countable compact base in the union topology.

3.10. Lemma. If $Y$ is locally compact and has a countable compact base then its one point compactification has a countable compact base.

4. Continuous images of Borel sets. In this section we study one-to-one projections of $K_{st}$ sets and continuous countable-to-one images of Borel sets. The main results are Theorems 4.6 and 4.7.

4.1. Lemma. If, for each $i \in \omega$, $Y_i$ is compact and $A_i$ is the one-to-one projection of a $K_{st}(X \times Y_i)$ onto $X$ then $\bigcap_{i \in \omega} A_i$ is the one-to-one projection of a $K_{st}\left(X \times \prod_{i \in \omega} Y_i\right)$ onto $X$.

Proof. For each $i \in \omega$, let $C_i \subseteq K_{st}(X \times Y_i)$ and $A_i$ be the one-to-one
projection of $C_i$ onto $X$. For $x \in A_i$, let $h_i(x)$ be the $y$ such that $(x, y) \in C_i$, $Z = \prod_{i \in \omega} Y_i$, and

$$F_i = \{(x, y) : x \in A_i, y \in Z \text{ and } y_i = h_i(x)\}.$$&

Then $F_i$ is homeomorphic to $C_i \times \prod_{j \in (\omega - \{i\})} Y_j$ so that $F_i \in K_{\sigma}(X \times Z)$. Let

$$D = \bigcap_{i \in \omega} F_i.$$&

Then $D \in K_{\sigma}(X \times Z)$ and $(x, y) \in D$ iff $x \in \bigcap_{i \in \omega} A_i$ and $y_i = h_i(x)$ for all $i \in \omega$. Thus, if $(x, y) \in D$ and $(x, y') \in D$ then $y_i = h_i(x) = y'_i$ for all $i \in \omega$ so that $y = y'$ and $\bigcap_{i \in \omega} A_i$ is the one-to-one projection of $D$ onto $X$.

4.2. **Lemma.** If, for each $i \in \omega$, $A_i$ is the one-to-one projection of a $K_{\sigma}(X \times Y_i)$ onto $X$, $A_i \cap A_j = 0 = Y_i \cap Y_j$ for $i \neq j$, and $Y' = \bigcup_{i \in \omega} Y_i$ with the union topology then $\bigcup_{i \in \omega} A_i$ is the one-to-one projection of a $K_{\sigma}(X \times Y')$ onto $X$.

**Proof.** Let $C_i \in K_{\sigma}(X \times Y_i)$, $A_i$ be the one-to-one projection of $C_i$ onto $X$, $D = \bigcup_{i \in \omega} C_i$. Then $\bigcup_{i \in \omega} A_i$ is the one-to-one projection of $D$. Moreover, $D$ is a $K_{\sigma}(X \times Y')$, for if

$$C_i = \bigcap_{j \in \omega} B(i, j) \quad \text{with } B(i, j) \in K_{\sigma}(X \times Y_i)$$

then for $i \neq i'$, $B(i, j) \cap B(i', k) = 0$ for all $j \in \omega$, $k \in \omega$ and hence

$$D = \bigcup_{i \in \omega} \bigcap_{j \in \omega} B(i, j) = \bigcap_{i \in \omega} \bigcup_{j \in \omega} B(i, j)$$

and

$$\bigcup_{i \in \omega} (B(i, i)) \in K_{\sigma}(X \times Y').$$

4.3. **Definition.** $A$ is a special set of uniqueness in $X$ iff there exist $Y$ and $C$ such that $Y$ is compact and has a countable compact base and $C \in K_{\sigma}(X \times Y)$ and $A$ is the one-to-one projection of $C$ onto $X$.

4.4. **Theorem.** If for each $i \in \omega$, $A_i$ is a special set of uniqueness in $X$ then $\bigcap_{i \in \omega} A_i$ is a special set of uniqueness in $X$ and if $A_i \cap A_j = 0$ for $i \neq j$ then $\bigcup_{i \in \omega} A_i$ is a special set of uniqueness in $X$.

**Proof.** $\bigcap_{i \in \omega} A_i$ is a special set of uniqueness in $X$ in view of 4.1 and 3.8. To see that if $A_i \cap A_j = 0$ for $i \neq j$ then $\bigcup_{i \in \omega} A_i$ is a special set of uniqueness in $X$, let $A_i$ be the one-to-one projection of $C_i$ where $C_i \in K_{\sigma}(X \times Y_i)$ and $Y_i$ is compact and has a countable compact
base. We may assume that $Y_i \cap Y_j = 0$ for $i \neq j$ for otherwise we may replace $Y_i$ by $Y_i \times \{ i \}$. Let $Y' = \bigcup_{i \in \omega} Y_i$ with the union topology and $Z$ be the one-point compactification of $Y'$. Then by Lemmas 3.9 and 3.10, $Z$ is compact and has a countable compact base. Moreover $K_\omega(X \times Y') \subset K_\omega(X \times Z)$. Hence by 4.2, $\bigcup_{i \in \omega} A_i$ is a special set of uniqueness in $X$.

4.5. Theorem. If $X$ has property I and $A \in \text{Borel } K(X)$ then $A$ is a special set of uniqueness in $X$.

Proof. If $A \in K_\omega(X)$ then clearly $A$ is a special set of uniqueness in $X$ since we can take $Y = \{ 0 \}$ and $A$ is the projection of $A \times \{ 0 \} \in K_\omega(X \times Y)$. Let $H$ be a maximal family such that $K(X) \subset H$ and if $A$ and $B$ are in $H$ then $A$ and $A - B$ are special sets of uniqueness in $X$. We shall show that $H$ is closed under countable unions and difference of two sets so that $\text{Borel } K(X) \subset H$. Let $A_i \in H$, for $i \in \omega$, and

$$S = \{ A : A \text{ is a special set of uniqueness in } X \}.$$

We now check, using 4.4:

(i) $A_0 - A_1 \in H$, for, $A_0 - A_1 \in S$ and for any $B \in H$,

$$(A_0 - A_1) - B = (A_0 - A_1) \cap (A_0 - B) \in S$$

and

$$B - (A_0 - A_1) = (B - A_0) \cup (B \cap A_0 \cap A_1) \in S;$$

(ii) $A_0 \cup A_1 \in H$ and hence

$$\bigcup_{i = 0}^n A_i \in H \quad \text{for } n \in \omega$$

for,

$$A_0 \cup A_1 = A_0 \cup (A_1 - A_0) \in S$$

and for any $B \in H$,

$$(A_0 \cup A_1) - B = (A_0 - B) \cup ((A_1 - B) \cap (A_1 - A_0)) \in S,$$

$$B - (A_0 \cup A_1) = (B - A_0) \cap (B - A_1) \in S;$$

(iii) $\bigcup_{i \in \omega} A_i \in H$, for, let

$$A_{n'} = A_n - \bigcup_{i = 0}^{n-1} A_i.$$

Then by (i) and (ii), $A_{n'} \in H$ and $A_{n'} \cap A_{k'} = 0$ for $n \neq k$. Hence
\[ \bigcup_{i \in \omega} A_i = \bigcup_{n \in \omega} A_n' \in S \]

and for any \( B \in H, \)

\[ \bigcup_{i \in \omega} A_i - B = \bigcup_{n \in \omega} (A_n' - B) \in S, \]

\[ B - \bigcup_{i \in \omega} A_i = \bigcap_{i \in \omega} (B - A_i) \in S. \]

4.6. Theorem. Let \( X \) have property I. Then \( A \) is Borel \( K(X) \) iff, for some \( B \in K_*(X), A \subseteq B \) and there exist \( X', C, f \) such that \( X' \) has property I, \( C \in K_*(X'), f \) is continuous and one-to-one on \( C \) and \( A = f(C) \).

Proof. If \( A \in Borel \ K(X) \) then by 4.5 \( A \) is the one-to-one projection of a \( K_*(X \times Y) \) for some \( Y \) that is compact and has a countable compact base. By 3.5, \( X \times Y \) has property I. Moreover, since \( X \) has property I, by 3.3 Borel \( K(X) = \) Borelian \( K(X) \) and hence \( A \subseteq B \) for some \( B \in K_*(X) \).

The converse is given by Theorem 6.3 in [2] again with the help of 3.3.

4.7. Theorem. If \( X \) has property I, \( A \) is Borel \( K(X) \), \( f \) is continuous and countable-to-one on \( A \) to some Hausdorff space \( Y \), and \( Y \in K_*(Y) \) then \( f(A) \) is Borelian \( K(Y) \) and \( f(A) \) has property I.

Proof. In view of 4.6, \( f(A) \) is the continuous countable-to-one image of a \( K_*(X') \) for some \( X' \) that has property I. Hence by Corollary 6.10 in [2], \( f(A) \) is Borelian \( K(Y) \). By 3.6, \( f(A) \) also has property I.

Bibliography


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